Relativity of arithmetics as a fundamental symmetry of physics

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Arithmetic operations can be defined in various ways, even if one assumes commutativity and associativity of addition and multiplication, and distributivity of multiplication with respect to addition. In consequence, whenever one encounters 'plus' or 'times' one has certain freedom of interpreting this operation. This leads to some freedom in definitions of derivatives, integrals and, thus, practically all equations occurring in natural sciences. A change of realization of arithmetics, without altering the remaining structures of a given equation, plays the same role as a symmetry transformation. An appropriate construction of arithmetics turns out to be particularly important for dynamical systems in fractal space-times. Simple examples from classical and quantum, relativistic and nonrelativistic physics are discussed.

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Symmetries of physical systems can be rather obvious or very abstract. Lorentz transformations, discovered as a formal symmetry of Maxwell's equations, seemed abstract until their physical meaning was understood by Einstein. Theory of group representations, the cornerstone of quantum mechanics and field theory, had its roots in Lie's studies of abstract symmetries of differential equations. It has taught us that differences in mathematical realizations of a symmetry may directly reflect physical differences.

Einstein's relativity, gauge invariance, Noether's theorems, Darboux-Bäcklund transformations, or supersymmetry are prominent examples of symmetry principles in physics. Here we discuss a new type of principle, occurring in any physical theory: The symmetry of mathematical equations under modifications of arithmetic operations, the induced modifications of derivatives and integrals included. Similarly to other physical symmetries, the symmetry maintains the form of relevant equations, but may possess different mathematical realizations. Fractal space-times provide nontrivial examples. A generalized arithmetics can lead to nontrivial continuous dynamics in sets of measure zero, invisible from the point of view of quantum mechanics. It opens a new room for phenomena such as dark energy, 'coming out of nowhere'.

To begin with, let us consider a bijection $f : X \to Y \subset \mathbb{R}$, where X is some set. The map f allows us to define addition, multiplication, subtraction, and division in X,

$$
\begin{aligned}
x \oplus y &= f^{-1}\big(f(x) + f(y)\big),\\
x \ominus y &= f^{-1}\big(f(x) - f(y)\big),\\
x \odot y &= f^{-1}\big(f(x)f(y)\big),\\
x \oslash y &= f^{-1}\big(f(x)/f(y)\big).
\end{aligned}
$$

One easily verifies the standard properties [1]: (1) associativity $(x\oplus y)\oplus z = x\oplus(y\oplus z)$, $(x\odot y)\odot z = x\odot(y\odot z)$, (2) commutativity $x \oplus y = y \oplus x$, $x \odot y = y \odot x$, (3) distributivity $(x \oplus y) \odot z = (x \odot z) \oplus (y \odot z)$. Elements $0, 1 \in X$ are defined by $0 \oplus x = x$, $1 \odot x = x$, which implies $f(0) = 0$, $f(1) = 1$. One further finds $x \ominus x = 0$, $x \oslash x = 1$, as expected [2]. In general, it is better to define subtraction independently of addition since it may happen that $f(-x)$ is undefined. An important nontrivial example of f is provided by the Cantor function [3], or more precisely the Cantor-line function defined below, where X is the Cantor subset of $[0, 1]$ and $Y = [0, 1]$. If $0 \ominus x$ exists, one can denote it by $\ominus x$.

Practically the only difference between $\oplus$, $\ominus$, $\odot$, $\oslash$ and $+$, $-$, $\cdot$, and $/$ is that in general multiplication is not just a repeated addition: Typically $x \oplus x \neq 2 \odot x$. Multiplication and addition are now truly independent.

Having all these arithmetic operations one can define a derivative of a function $A : X \to X$,

$$
\frac{d_f A(x)}{d_f x} = \lim_{h\to 0}\Big(A(x \oplus h) \ominus A(x)\Big) \oslash h, \qquad (1)
$$

satisfying

$$
\begin{aligned}
\frac{d_f A(x) \odot B(x)}{d_f x} &= \frac{d_f A(x)}{d_f x} \odot B(x) \oplus A(x) \odot \frac{d_f B(x)}{d_f x},\\
\frac{d_f A(x) \oplus B(x)}{d_f x} &= \frac{d_f A(x)}{d_f x} \oplus \frac{d_f B(x)}{d_f x},\\
\frac{d_f A[B(x)]}{d_f x} &= \frac{d_f A[B(x)]}{d_f B(x)} \odot \frac{d_f B(x)}{d_f x}.
\end{aligned}
$$

Now consider functions $F : Y \to Y$ and $F_f : X \to X$ related by

$$
F_f(x) = f^{-1}\Big(F\big(f(x)\big)\Big). \qquad (2)
$$

Employing (1) and the fact that $f(0) = 0$ one finds

$$
\frac{d_f F_f(x)}{d_f x} = f^{-1}\Big(F'\big(f(x)\big)\Big), \qquad (3)
$$

where $F'(y) = dF/dy$ is the usual derivative in Y, defined in terms of $+$, $-$, $\cdot$, and $/$. It is extremely important to note that (3) has been derived with no need of differentiability of f. $f(0) = 0$ is enough to obtain a well defined derivative. (3) is not the standard formula known for composite functions since no derivatives of f occur. To understand why functions of the form (2) are so essential let us solve the differential equation

$$\frac{d_f A(x)}{d_f x} = A(x), \quad A(0) = 1$$

by assuming that $A(x) = \oplus_{n=0}^{\infty} a_n \odot x^{\odot n}$, where $x^{\odot n} = x \odot \cdots \odot x$ (n times). Then, comparing term by term, one finds the unique solution

$$A(x) = f^{-1}\big(e^{f(x)}\big) = \exp_f x,$$

fulfilling $\exp_f(x \oplus y) = \exp_f x \odot \exp_f y$. Its inverse is $\ln_f x = f^{-1}\big(\ln f(x)\big)$, $\ln_f(x \odot y) = \ln_f x \oplus \ln_f y$.

As our next example consider a classical harmonic oscillator

$$\frac{d_f^2 x(t)}{d_f t^2} = \frac{d_f}{d_f t}\frac{d_f x(t)}{d_f t} = \ominus \omega^{\odot 2} \odot x(t)$$

where $\omega^{\odot 2} = \omega \odot \omega$. The minus sign has to have a precise meaning so here we assume that $-f(x) = f(-x)$. Setting $x(t) = \oplus_{n=0}^{\infty} a_n \odot t^{\odot n}$, one obtains

$$x(t) = C_1 \odot \sin_f(\omega \odot t) \oplus C_2 \odot \cos_f(\omega \odot t)$$

where

$$\sin_f x = f^{-1}\big(\sin f(x)\big), \quad \cos_f x = f^{-1}\big(\cos f(x)\big),$$

and C_1, C_2 are constants.

An instructive exercise is to plot phase-space trajectories of the harmonic oscillator corresponding to various choices of f. Fig. 1 shows the trajectories for the Cantor-line function, defined below, and $f(x) = x^n$, with $n = 1, 3, 5$. All these trajectories represent a classical harmonic oscillator that satisfies the usual law of 'force oppositely proportional to displacement', with conserved energy '$\dot{x}^2 + \omega^2 x^2$', but with different meanings of 'plus' and 'times'. The resulting trigonometric functions are essentially the chirp signals [4] known from signal analysis.

One might still have the impression that what we do is just standard physics in nonstandard coordinates. So, consider the problem of a fractal Universe of dimension $4 - \epsilon$, analogous to the one arising in causal dynamical triangulation theory [5]. Our physical equations have to be formulated in terms of notions that are intrinsic to the Universe, but what should be meant by a velocity, say? We have to subtract positions and divide by time, but we have to do it in a way that is intrinsic to the Universe we live in. Moreover, from our perspective positions and flow of time seem continuous even if they would appear discontinuous from an exactly 4-dimensional perspective. We should not make the usual step and turn to fractional derivatives [6], since for inhabitants of $(4-\epsilon)$-dimensional Universe the velocity is just the first derivative of position with respect to time, and not some derivative of order $0 < \alpha < 1$.

As usual, Cantor-like sets and Cantor-type functions provide a rich source of highly nontrivial examples [7, 8]. A simple model of $(4 - \epsilon)$-dimensional space-time is the Cartesian product of four Cantor dusts, appropriately extended to the whole of $\mathbb{R}$. Cantor dusts are easy to work with and possess certain mathematical universality [9], but can be defined in different ways. Here we need a precisely constructed fractal X and a bijection $f : X \to Y$. Let us start with the right-open interval $[0, 1) \subset \mathbb{R}$, and let the (countable) set $Y_2 \subset [0, 1)$ consist of those numbers that have two different binary representations. Denote by $0.t_1t_2\ldots$ a ternary representation of some $x \in [0, 1)$. If $y \in Y_1 = [0, 1) \setminus Y_2$ then y has a unique binary representation, say $y = 0.b_1b_2\ldots$. One then sets $f^{-1}(y) = 0.t_1t_2\ldots$, $t_j = 2b_j$. Let $y = 0.b_1b_2\cdots = 0.b'_1b'_2\ldots$ be the two representations of $y \in Y_2$. We define $f^{-1}(y) = \min\{0.t_1t_2\ldots, 0.t'_1t'_2\ldots\}$, where $t_j = 2b_j$, $t'_j = 2b'_j$. We have therefore constructed an injective map $f^{-1} : [0, 1) \to [0, 1)$. The triadic Cantor-like set is defined as the image $C_{[0,1)} = f^{-1}\big([0, 1)\big)$, and $f : C_{[0,1)} \to [0, 1)$, $f = (f^{-1})^{-1}$, is the required bijection between $C_{[0,1)}$ and the interval. For example, $1/2 \in Y_2$ since $1/2 = 0.1_2 = 0.0(1)_2$. We find $f^{-1}(1/2) = \min\{0.2_3 = 2/3, 0.0(2)_3 = 1/3\} = 1/3$. Accordingly, $1/3 \in C_{[0,1)}$ while $2/3 \notin C_{[0,1)}$. $C_{[0,1)}$ is not exactly the standard Cantor set, but all irrational elements of the Cantor set belong to $C_{[0,1)}$ (an irrational number has a unique binary form), together with some rational numbers such as $1/3$. Note further that $0 \in C_{[0,1)}$, with $f(0) = 0$. We could proceed analogously with $1 \notin [0, 1)$, since $1 = 1.(0)_2 = 0.(1)_2$ possesses two binary representations with $\min\{2.(0)_3, 0.(2)_3\} = \min\{2, 1\} = 1$. However, instead of including 1 in $C_{[0,1)}$, let us shift $C_{[0,1)}$ to the right by 1, thus obtaining $C_{[1,2)}$. Proceeding in this way we construct a fractal $C = \cup_{k\in\mathbb{Z}} C_{[k,k+1)}$, and the bijection $f : C \to \mathbb{R}$. Explicitly, if $x \in C_{[0,1)}$, then $x + k \in C_{[k,k+1)}$, and $f(x + k) = f(x) + k$ by definition. Let us call C the Cantor line, and f the Cantor-line function.

An integral is defined so that the fundamental laws of calculus,

$$\int_a^b \frac{d_f A(x)}{d_f x} \odot d_f x = A(b) \ominus A(a),$$

and

$$\frac{d_f}{d_f x} \int_a^x A(x') \odot d_f x' = A(x), \tag{4}$$

hold true. The explicit form reads

$$\int_a^b F_f(x) \odot d_f x = f^{-1}\left(\int_{f(a)}^{f(b)} F(y) dy\right), \tag{5}$$

where $\int F(y)dy$ is the standard (say, Lebesgue) integral in $\mathbb{R}$.

The integral so defined is not equivalent to the fractal measure. Indeed, the fractal measure of the Cantor set

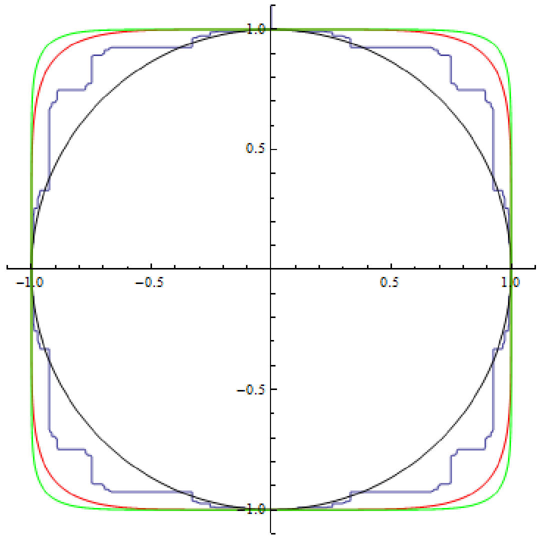

FIG. 1: Phase-space trajectories of the harmonic oscillator with $\omega = 1$ and $f(x) = x$ (black), $f(x) = x^3$ (red), $f(x) = x^5$ (green), and the Cantor-line function (blue). Taking $f(x) = x^n$ with sufficiently large n we would find a dynamics looking like a motion along a square.

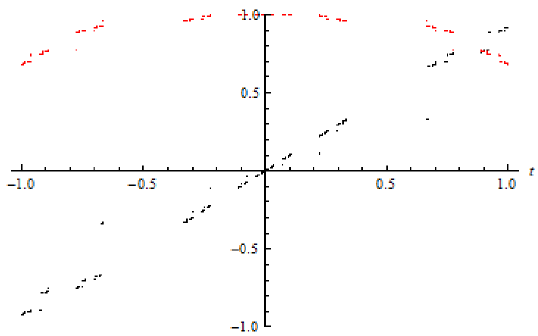

FIG. 2: Cantor-world oscillation in Cantorian time. $\sin_f(t)$ (black) and $\cos_f(t)$ (red) for the Cantor-line function f. Inhabitants of the Cantor-dust space-time would experience this as a continuous process.

embedded in an interval of length L is L^D, where $D = \log_3 2$. Thus, for $L = 1/3$ one finds $L^D = 1/2$. Since segments $[0, 1/3]$ and $[2/3, 1]$ both have $L = 1/3$ they both have the same D-dimensional volume equal $1/2$. Taking $F_f(x) = 1$ we find

$$\int_a^b d_f x = \int_a^b \frac{d_f x}{d_f x} \odot d_f x = f^{-1}\big(f(b) - f(a)\big),$$

and $\int_0^{1/3} d_f x = 1/3$, $\int_{1/3}^1 d_f x = 1/3$, $\int_0^1 d_f x = (1/3) \oplus (1/3) = 1$.

Now let us switch to higher dimensional examples. First consider the plane, i.e. the Cartesian product of two lines. One checks that $\sin_f^{\odot 2} x \oplus \cos_f^{\odot 2} x = 1$, $\cosh_f^{\odot 2} x \ominus \sinh_f^{\odot 2} x = 1$. It is an appropriate place to stress that our approach does not seem to be related to exponential, trigonometric, and hyperbolic functions defined (in various ways) in the context of time-scales dynamics [12–14], or in non-extensive thermodynamics [15, 16], but there are links to Kolmogorov-Nagumo averages employed in Rényi information theory [16, 19] (see below). Functions $\sin_f$, $\cos_f$, $\sinh_f$, $\cosh_f$, satisfy the basic standard formulas such as

$$\sin_f(a \oplus b) = \sin_f a \odot \cos_f b \oplus \cos_f a \odot \sin_f b$$

and the like, so

$$\begin{aligned} x' &= x \odot \cos_f \alpha \oplus y \odot \sin_f \alpha, \\ y' &= y \odot \cos_f \alpha \ominus x \odot \sin_f \alpha, \end{aligned}$$

defines a rotation. The rotation satisfies the usual group composition rule, a fact immediately implying that one can work with generalized-arithmetics matrix equations. In an analogous way one arrives at Lorentz transformations in Cantorian Minkowski space, the Cartesian product of four Cantor lines with the invariant form

$$x_\mu \odot x^\mu = x_0^{\odot 2} \ominus x_1^{\odot 2} \ominus x_2^{\odot 2} \ominus x_3^{\odot 2}.$$

Such Lorentz transformations are unrelated to those occurring in exactly 4-dimensional fractal space-time of scale relativity [10]. In $1+1$ dimensions, employing,

$$\begin{aligned} x'^0 &= x^0 \odot \cosh_f \alpha \ominus x^1 \odot \sinh_f \alpha, \\ x'^1 &= \ominus x^0 \odot \sinh_f \alpha \oplus x^1 \odot \cosh_f \alpha, \end{aligned}$$

and $x'^1 = 0$, one finds

$$\beta = x^1 \oslash x^0 = \tanh_f \alpha.$$

In consequence, the fact that $f(1) = 1$ sets the limit $|\beta| \le 1$ for maximal velocity independently of the choice of f. In principle, problems such as clock synchronization, composition of velocities, or the twin paradox, may lead to direct experimental tests of f and some insights into a putative fractal structure of space-time.

Arithmetics of complex numbers requires some care. One should not just take $f : \mathbb{C} \to \mathbb{C}$ due to the typical multi-valuedness of f^{-1} and the resulting ill-definiteness of $\oplus$ and $\odot$. Definition of i as a $\pi/2$ rotation also does not properly work since one cannot guarantee a correct behaviour of $i^{\odot n}$ for a general f. The correct solution is the simplest one: One should treat complex numbers as pairs of reals satisfying the following arithmetics

$$\begin{aligned} (x, y) \oplus (x', y') &= (x \oplus x', y \oplus y'), \\ (x, y) \odot (x', y') &= (x \odot x' \ominus y \odot y', y \odot x' \oplus x \odot y'), \\ i &= (0, 1), \end{aligned}$$

supplemented by conjugation $(x, y)^* = (x, -y)$. As stressed in [11], the resulting complex structure is just the standard one, but no mysterious 'imaginary number' is employed.

In this way we have arrived at quantum mechanics. As our final example let us solve the eigenvalue problem for a 1-dimensional harmonic oscillator. Consider

$$\begin{aligned}\hat{H}_f \psi_f(x) &= -\alpha^{\odot 2} \odot \frac{d_f^2 \psi_f(x)}{d_f^2 x} \oplus \beta^{\odot 2} \odot x^{\odot 2} \odot \psi_f(x) \\ &= E_f \odot \psi_f(x),\end{aligned}$$

where α, β are parameters. The normalized ground state is

$$\psi_{0f}(x) = f^{-1}\left(\left(\frac{f(\beta)}{\pi f(\alpha)} \right)^{1/4} e^{-\frac{f(\beta) f(x)^2}{2f(\alpha)}} \right),$$

with the eigenvalue $E_{0f} = \alpha \odot \beta$. The excited states can be derived in the usual way.

There are two peculiarities of the resulting quantum mechanics one should be aware of. First of all, if f is a Cantor-like function representing a fractal whose dimension is less that 1, then the real-line Lebesgue measure of the fractal is zero. Keeping in mind that states in quantum mechanics are represented by equivalence classes of wave functions that are identical up to sets of measure zero, we can remove the Cantor line from $\mathbb{R}$ without altering standard quantum mechanics. Having removed the Cantor line C from $\mathbb{R}$ we still can do ordinary quantum mechanics on $\mathbb{R} \setminus C$, whereas C itself can become a universe for its own, Cantorian theory. Removing C from $\mathbb{R}$ does not mean that we impose some fractal-like boundary conditions or consider a Schrödinger equation with a delta-peaked potential of Cantor-set support [21]. We just use the freedom to modify wave functions on sets of measure zero. So we can keep the standard Gaussian $f(x) = x$ ground state on $\mathbb{R} \setminus C$, and employ the Cantorian $\psi_{0f}(x)$ on C. According to quantum mechanics the resulting wave function belongs to the same equivalence class as the usual Gaussian, and thus represents the same state. However, now the energy is $\hbar\omega/2 + \alpha \odot \beta$, with $\alpha \odot \beta$ 'appearing from nowhere'. The analogy to dark energy is evident. The additional energy is a real number so it can be added to $\hbar\omega/2$, similarly to many other energies that occur in physics and are additive in spite of unrelated origins.

The second subtlety concerns physical dimensions of various quantities occurring in f-generalized arithmetics. Even the simple case of $\omega \odot t$ may imply a necessity of dimensionless ω and t if f is sufficiently nontrivial (functions of the form $f(x) = x^q$, $q \in \mathbb{R}_+$, are in this respect exceptional since then $x \odot y = xy$). In general we have to work with dimensionless variables x in order to make $f(x)$ meaningful. It is thus simplest to begin with reformulating all the 'standard' theories in dimensionless forms, similarly to $c = 1$ and $\hbar = 1$ conventions often employed in relativity and quantum theory.

Quantum mechanics has brought us to the issue of probability. An appropriate normalization is $\oplus_k p_k = 1$ which, in virtue of $f(1) = 1$, implies $\sum_k f(p_k) = \sum_k P_k = 1$. We automatically obtain two coexisting but inequivalent sets of probabilities, in close analogy to probabilities P_k and escort probabilities $p_k = P_k^q$ occurring in generalized statistics and multifractal theory [15, 16]. Averages

$$\langle a \rangle_f = \oplus_k p_k \odot a_k = f^{-1}\Big(\sum_k P_k f(a_k) \Big),$$

have the form of Kolmogorov-Nagumo averages [16–18], which implies the usual bounds $a_{\min} \le \langle a \rangle_f \le a_{\max}$. From the point of view of modified arithmetics the constraints one should impose on escort probabilities and Kolmogorov-Nagumo averages are, though, completely different from those employed in nonextensive statistics and Rényi's information theory [19], provided instead of additivity one has $\oplus$-additivity in mind. Rényi's

$$f(x) = \frac{2^{(1-q)x} - 1}{2^{1-q} - 1}$$

can be replaced by a much wider class of fs. Of some interest is the analogue

$$\big| \langle AB \rangle_f \oplus \langle AB' \rangle_f \oplus \langle A'B \rangle_f \ominus \langle A'B' \rangle_f \big| \le f^{-1}(2),$$

of the CHSH inequality [20] since, in general, $f^{-1}(2) \neq 2$. Incidentally, for our choice of the Cantor-line function, with $f(x + k) = f(x) + k$, $k \in \mathbb{Z}$, one finds the standard CHSH bound $f^{-1}(2) = 2$.

The modified calculus is as simple as the one one knows from undergraduate education. What may be nontrivial is to find f if X is a sufficiently 'strange' object. The case of the Cantor line was relatively obvious, but the choice of f may be much less evident if X is a multifractal or a higher-dimensional fractal.

In order to conclude, let us return to Fig. 1. All the phase-space trajectories represent the *same* physical system: A harmonic oscillator satisfying the Newton equation $d^2x/dt^2 = -\omega^2 x$, with the same physical parameters for each of the trajectories. So how come the trajectories are different? The answer is: Because the very form of Newton's equation does not tell us what should be meant by 'plus' or 'times'. This observation extends to any theory that employs arithmetics of real numbers. It would not be very surprising if some alternative arithmetics proved essential for Planck-scale physics, where fractal space-time is expected [22], or to biological modeling where fractal structures are ubiquitous. Links of f-generalized trigonometric functions to chirps suggest that fs even much more 'ordinary' than fractal functions may also lead to nontrivial applications.

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[1] Detailed proofs can be found in the preprint version of the paper, see arXiv:1412.8583v2 [math-ph].
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