MATHEMATICA APPLICANDA
Vol. 41(2) 2013, p. 185–198
doi: 10.14708/ma.v41i2.466

Tomasz Szarek* (Gdańsk)

Lower bound technique and its applications to function systems and stochastic partial differential equations

Abstract We formulate some criteria for the existence of an invariant measure for Markov chains and Markov processes. We also show their application in the theory of function systems and stochastic differential equations

2000 Mathematics Subject Classification: Primary 60J25, Secondary 60G51.

Key words and phrases: Ergodicity of Markov families, invariant measures, function systems, stochastic differential equations.

1. Introduction We aim to present some ideas from the lower bound technique. The lower bound technique is a useful tool in the ergodic theory of Markov processes. It has been used by Doeblin, see [4], to show mixing of a Markov chain whose transition probabilities possess a uniform lower bound. A somewhat different approach, relying on the analysis of the operator dual to the transition probability, has been applied by A. Lasota and J. Yorke, see e.g. [11]. For example in [13] they show that the existence of a lower bound for the iterates of the Frobenius–Perron operator, which corresponds to a piecewise monotonic transformation of the unit interval, implies the existence of a stationary distribution for the deterministic Markov chain describing the iterates of the transformation. In fact, the invariant measure is then unique in the class of measures that are absolutely continuous with respect to the one dimensional Lebesgue measure, and statistically stable, i.e. the law of the chain, starting with any initial distribution that is absolutely continuous, converges to an invariant measure in the total variation metric. This technique has been extended to more general Markov chains, including those corresponding to iterated function systems, see e.g. [14]. However, most of

* Publication co-financed by the European Union as part of the European Social Fund within the project Center for Applications of Mathematics. This research was partly supported by the National Science Centre of Poland, grant number DEC-2012/07/B/ST1/03320

the existing results are formulated for Markov chains taking values in some finite dimensional spaces, see e.g. [22] for a review of the topic. This assumption is very restrictive since many applications we have in mind are defined in infinitely dimensional spaces (e.g. function spaces in the case of stochastic partial differential equations).

Generally speaking, the lower bound technique consists of deriving ergodic properties of the Markov process from the fact that there exists a "small" set in the state space, e.g. it could be compact, such that the time averages of the mass of the process are concentrated on that set for all sufficiently large times. If this set is compact, one can conclude the existence of an invariant probability measure with not much difficulty, e.g. by the so-called Krylov–Bogolubov theorem (see [11]).

The question of extending the lower bound technique to Markov processes taking values in Polish spaces that are not locally compact is quite a delicate matter. We stress here that to prove the existence of a stationary measure, it is not sufficient to derive a lower bound on the transition probability over some "thin" set. One can show, see the counterexample provided in [16], that even if the mass of the process, contained in any neighbourhood of a given point, is separated from zero for all times, an invariant measure may fail to exist. Therefore, we have introduced the concept of so-called e–chains and e–processes (see [12,16]). A Markov process $(\Phi_t)_{t\geqslant 0}$ on some Polish space (X,ρ) with the corresponding Markov semigroup $(\pi^t)_{t\geqslant 0}$ satisfies the *e–property* (and consequently is an e–process) if for any bounded Lipschitz function $\psi : X \to \mathbb{R}$ the family $(U^t\psi)_{t\geqslant 0}$ corresponding to $(\Phi_t)_{t\geqslant 0}$ is equicontinuous.

In literature we may find a lot of processes with this property that have been intensively studied recently (see for instance [7–9, 18, 20, 21]). For e–processes we are able to formulate criteria for the existence of an invariant measure and its stability. These criteria will be presented in Section 2. Section 3 presents some applications of these general results in the theory of iterated function systems (**IFS**'s) and stochastic partial differential equations (**SPDE's**).

2. Markov operators and semigroups of Markov operators

Let (X,ρ) be a complete and separable metric space and let $\mathbf{\Phi} = (\Phi_n)_{n\geqslant 1}$ be a discrete–time Markov chain on X. By $\mathcal{B}(X)$ we denote the space of all Borel sets. Let $\pi(x, A)$ be a transition function defined for $x \in X$ and $A \in \mathcal{B}(X)$. **Feller's property** means that the function $x \to \pi(x, O)$ is lower semicontinuous for all open sets O. Alternatively, we can say that

$$C(X) \ni f(\cdot) \to Uf(\cdot) = \int_X f(y)\pi(\cdot, dy) \in C(X),$$

where $C(X)$ denotes the space of all bounded continuous functions on X equipped with the supremum norm.

We are interested in the existence of an invariant probability measure for $\mathbf{\Phi}$. A Borel measure μ_* is called **invariant** if

$$\mu(A) = P\mu(A) = \int_X \pi(x, A)\mu(dx)$$

for $A \in \mathcal{B}(X)$.

By $\mathcal{M}$ and $\mathcal{M}_1$ we shall denote the space of all bounded Borel measures and normed Borel measures, respectively. Let μ be an arbitrary Borel measure. We define the **support of the measure** $\mu \in \mathcal{M}$ by setting

$$\operatorname{supp} \mu = \{x \in X : \mu(B(x, \varepsilon)) > 0 \text{ for every } \varepsilon > 0\}.$$

Let $\mathcal{L} \subset \mathcal{M}_1$. The family $\mathcal{L}$ is called **tight** if for any $\varepsilon > 0$ there is a compact set $K \subset X$ such that

$$\mu(K) \geqslant 1 - \varepsilon \qquad \text{for all } \mu \in \mathcal{L}.$$

In order to establish the existence of an invariant measure and stability, we introduce the following condition:

($\mathcal{E}$) There exists $z \in X$ such that for every open set O containing z

$$\limsup_{n\to\infty} \left(\frac{1}{n} \sum_{i=1}^{n} \pi^i(x, O) \right) > 0 \qquad \text{for some } x \in X. \tag{2.1}$$

Here π^i denotes the i-th iterate of the transition function π, i.e.

$$\pi^2(x, A) = \int_X \pi(x, \mathrm{d}y)\pi(y, A) \quad \text{and} \quad \pi^{n+1}(x, A) = \int_X \pi^n(x, \mathrm{d}y)\pi(y, A) \text{ for } n \geqslant 1,$$

and $x \in X$ and $A \in \mathcal{B}(X)$.

Theorem 2.1 **(ref. [16])** Let $\pi\colon X \times \mathcal{B}(X) \to [0, 1]$ be a transition function for a discrete–time Markov chain $\mathbf{\Phi}$ with the Feller property and assume that condition ($\mathcal{E}$) holds for some $z \in X$. If $\{U^n f \colon n \in \mathbb{N}\}$ is equicontinuous in z for every Lipschitz continuous function f, then $\mathbf{\Phi}$ admits an invariant probability measure.

Since the proof is very characteristic to proofs we meet using the lower bound technique, we provide it here.

Proof It suffices to show that for every $\varepsilon > 0$ there exists a compact set $K \subset X$ such that

$$\liminf_{n\to\infty} \pi^n(z, K^\varepsilon) \geqslant 1 - \varepsilon, \tag{2.2}$$

where $K^\varepsilon = \{x \in X : \inf_{y\in K} \rho(x, y) < \varepsilon\}$. This, in conjunction with Theorem 2.2 in [6], tells us that the measures $\{\pi^n(z, \cdot) : n \in \mathbb{N}\}$ are tight. Therefore, the Cesaro averages are weakly precompact by the Prokhorov theorem (see [6]). Note that any weak limit of the Cesaro averages is invariant.

Assume, contrary to our claim, that (2.2) does not hold for some $\varepsilon > 0$. By Ulam's lemma (see [2]) there exist a sequence of compact sets $(K_i)_{i\geqslant1}$ and a sequence of integers $(q_i)_{i\geqslant1}$ satisfying

$$\pi^{q_i}(z, K_i) > \varepsilon$$

and

$$\min\{\rho(x, y) : x \in K_i, y \in K_j\} \geqslant \varepsilon/3 \quad \text{for } i, j \in \mathbb{N},\ i \neq j. \tag{2.3}$$

We first show that for every open set O containing z and $j \in \mathbb{N}$ there exist $y \in O$ and $i \geqslant j$ such that

$$\pi^{q_i}(y, K_i^{\varepsilon/12}) < \varepsilon/2.$$

On the contrary, suppose that there exist an open set O' containing z and $i_0 \in \mathbb{N}$ such that

$$\inf\{\pi^{q_i}(y, K_i^{\varepsilon/12}) : y \in O',\ i \geqslant i_0\} \geqslant \varepsilon/2. \tag{2.4}$$

Let $x \in X$ be such that condition (2.1) holds with O' in place of O. Let $\alpha > 0$ be such that

$$\limsup_{n\to\infty} \left(\frac{1}{n}\sum_{i=1}^{n} \pi^i(x, O')\right) > \alpha.$$

By (2.3), (2.4) and the Chapman–Kolmogorov equation we obtain

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \pi^i\left(x, \bigcup_{j=i_0}^{N} K_j^{\varepsilon/12}\right) > (N - i_0)\alpha\varepsilon/2$$

for every $N \geqslant i_0$, which is impossible.

Now we will define by induction a sequence of Lipschitz continuous functions $(\tilde{f}_n)_{n\geqslant1}$, a sequence of points $(y_n)_{n\geqslant1}$, $y_n \to z$ as $n \to \infty$, and three increasing sequences of integers $(i_n)_{n\geqslant1}$, $(k_n)_{n\geqslant1}$, $(m_n)_{n\geqslant1}$, $i_{n+1} > k_n > i_n$ for $n \in \mathbb{N}$, such that

$$\tilde{f}_{n|K_{i_n}} = 1 \quad \text{and} \quad 0 \leqslant \tilde{f}_n \leqslant \mathbf{1}_{K_{i_n}^{\varepsilon/12}}, \tag{2.5}$$

$$|U^{m_n}(\sum_{i=1}^{n} \tilde{f}_i)(z) - U^{m_n}(\sum_{i=1}^{n} \tilde{f}_i)(y_n)| > \varepsilon/4 \tag{2.6}$$

and

$$\pi^{m_n}\left(u, \bigcup_{i=k_n}^{\infty} K_i^{\varepsilon/12}\right) < \varepsilon/16 \quad \text{for } u = z,\ y_n,\quad n \in \mathbb{N}. \tag{2.7}$$

Let $n = 1$. From what has already been proved, it follows that there exist $y_1 \in B(z,1)$ and $i_1 \in \mathbb{N}$ such that

$$\pi^{q_{i_1}}(y_1, K_{i_1}^{\varepsilon/12}) < \varepsilon/2.$$

Set $m_1 = q_{i_1}$ and let $k_1 > i_1$ be such that

$$\pi^{m_1}\left(u, \bigcup_{i=k_1}^{\infty} K_i^{\varepsilon/12}\right) < \varepsilon/16 \quad \text{for } u = z,\ y_1.$$

Let $\tilde{f}_1$ be an arbitrary Lipschitz function satisfying

$$\tilde{f}_{1|K_{i_1}} = 1 \quad \text{and} \quad 0 \leqslant \tilde{f}_1 \leqslant \mathbf{1}_{K_{i_1}^{\varepsilon/12}}. \tag{2.8}$$

Thus

$$|U^{m_1}\tilde{f}_1(z) - U^{m_1}\tilde{f}_1(y_1)| \geqslant P^{m_1}(z, K_{i_1}) - P^{m_1}(y_1, K_{i_1}^{\varepsilon/12}) > \varepsilon/2.$$

If $n \geqslant 2$ is fixed and $\tilde{f}_1, \ldots, \tilde{f}_{n-1};\ y_1, \ldots, y_{n-1};\ i_1, \ldots, i_{n-1};\ k_1, \ldots, k_{n-1};\ m_1, \ldots, m_{n-1}$ are given, we choose $\sigma < n^{-1}$ such that

$$\left| U^m\left(\sum_{i=1}^{n-1} \tilde{f}_i\right)(z) - U^m\left(\sum_{i=1}^{n-1} \tilde{f}_i\right)(y) \right| < \varepsilon/8 \tag{2.9}$$

for $y \in B(z,\sigma)$ and $m \in \mathbb{N}$. Similarly, as in the first part, we may choose $y_n \in B(z,\sigma)$ and $i_n > k_{n-1}$ such that

$$\pi^{q_{i_n}}(y_n, K_{i_n}^{\varepsilon/12}) < \varepsilon/2.$$

Set $m_n = q_{i_n}$ and let $\tilde{f}_n$ be an arbitrary Lipschitz function satisfying condition (2.5). Let $k_n > i_n$ be such that

$$\pi^{m_n}\left(u, \bigcup_{i=k_n}^{\infty} K_i^{\varepsilon/12}\right) < \varepsilon/16 \quad \text{for } u = z,\ y_n.$$

From this, (2.9) and the definition of $\tilde{f}_n$ we have

$$\begin{aligned}
&\left| U^{m_n}\left(\sum_{i=1}^{n} \tilde{f}_i\right)(z) - U^{m_n}\left(\sum_{i=1}^{n} \tilde{f}_i\right)(y_n) \right| \\
&\geqslant \left| U^{m_n}\tilde{f}_n(z) - U^{m_n}\tilde{f}_n(y_n) \right| \\
&- \left| U^{m_n}\left(\sum_{i=1}^{n-1} \tilde{f}_i\right)(z) - U^{m_n}\left(\sum_{i=1}^{n-1} \tilde{f}_n\right)(y_n) \right| \\
&> \varepsilon/2 - \varepsilon/8 > \varepsilon/4.
\end{aligned}$$

We now define $f = \sum_{i=1}^{\infty} \tilde{f}_i$. By (2.3) and (2.5) f is a Lipschitz continuous function and $\|f\|_\infty \leqslant 1$. Finally, by (2.6) and (2.7) we have

$$|U^{m_n} f(z) - U^{m_n} f(y_n)| > \varepsilon/8 \qquad \text{for } n \in \mathbb{N}$$

and since $y_n \to z$ as $n \to \infty$, this contradicts the assumption that $\{U^n f \colon n \in \mathbb{N}\}$ is equicontinuous in z. ■

Let $\mathbf{\Phi} = (\Phi_t)_{t\geqslant 0}$ be a time–homogeneous Markov process on X. Let $(\pi^t)_{t\geqslant 0}$ be the semigroup of transition functions corresponding to $\mathbf{\Phi}$. Analogously to the discrete case, we say that $\mathbf{\Phi}$ satisfies the **Feller property** if the function

$$x \to \pi^t(x, O)$$

is lower semicontinuous for all open sets O and $t \geqslant 0$. Alternatively, we can say that the semigroup $(U^t)_{t\geqslant 0}$ satisfies $U^t f \in C(X)$ for $f \in C(X)$, $t \geqslant 0$, where

$$U^t f(x) = \int_X f(y)\pi^t(x, dy) \quad \text{for } x \in X \text{ and } t \geqslant 0.$$

Let $(P^t)_{t\geqslant 0}$ denote the Markov semigroup on $\mathcal{M}$ (dual to $(U^t)_{t\geqslant 0}$) given by the formula:

$$P^t\mu(A) = \int_X U^t \mathbf{1}_A(y)\mu(\mathrm{d}y) \quad \text{for } A \in \mathcal{B}(X),\ \mu \in \mathcal{M} \text{ and } t \geqslant 0.$$

Obviously $P^t\delta_x(A) = \pi^t(x, A)$ for $x \in X, A \in \mathcal{B}(X)$ and $t \geqslant 0$. Therefore, when it does not lead to a contradiction, $(P^t)_{t\geqslant 0}$ will be also called the transition semigroup.

A measure $\mu_* \in \mathcal{M}$ is called **invariant** for $\mathbf{\Phi}$ if $P^t\mu_* = \mu_*$ for all $t \geqslant 0$. By **"w-lim"** we shall denote the limit in the sense of weak convergence of measures. We say that $\mathbf{\Phi}$ is asymptotically stable if there exists an invariant measure $\mu_* \in \mathcal{M}_1$ such that

$$\operatorname*{w-lim}_{t\to\infty} P^t\nu = \mu_* \quad \text{for any } \nu \in \mathcal{M}_1.$$

The family $(U^t)_{t\geqslant 0}$ is called **equicontinuous** if for every bounded Lipschitzian function f, the family of functions $\{U^t f : t \geqslant 0\}$ is equicontinuous on compact sets.

In order to establish the existence of an invariant measure, we may introduce the following condition:

$(\mathcal{E}_1)$ There exists a compact set $F \subset X$ having the following property: for every open neighbourhood U of F there exists $x \in X$ such that

$$\limsup_{t\to\infty} \left(\frac{1}{t} \int_0^t \pi^s(x, U)\mathrm{d}s \right) > 0. \tag{2.10}$$

The next theorem is an extension of Theorem 2.1 to continuous–time Markov processes.

Theorem 2.2 **(ref. [12])** Let $(\pi^t)_{t\geqslant 0}$ be the semigroup of transition functions corresponding to a Markov process $\mathbf{\Phi}$ which satisfies the Feller property. Assume that condition $(\mathcal{E}_1)$ holds. If $(U^t)_{t\geqslant 0}$ is equicontinuous, then $\mathbf{\Phi}$ admits an invariant probability measure.

For a given $t > 0$ and $\mu \in \mathcal{M}_1$ define $Q^t\mu := t^{-1}\int_0^t P^s\mu ds$. When $t = 0$, we adopt the convention $Q^0\mu := \mu$. We also write $Q^t(x, \cdot)$ in the particular case when $\mu = \delta_x$. Let

$$\mathcal{T} := \left\{ x \in X : \text{the family of measures } \Big(Q^t(x,\cdot)\Big)_{t\geqslant 0} \text{ is tight } \right\}. \tag{2.11}$$

In [10] we proved that if the family $(U^t)_{t\geqslant 0}$ is equicontinuous, then any invariant measure μ_* satisfies $\operatorname{supp}\mu_* \subset \mathcal{T}$.

In the same spirit as in the case of Theorem 2.3, we prove the following theorem.

Theorem 2.3 **(ref. [10, 16])** Let $(\pi^t)_{t\geqslant 0}$ be the semigroup of transition functions corresponding to a Markov process $\mathbf{\Phi}$ which satisfies the Feller property. Let $(U^t)_{t\geqslant 0}$ be equicontinuous. Assume that condition $(\mathcal{E}_1)$ holds with $F = \{z\}$ and let $\mu_* \in \mathcal{M}_1$ be its invariant measure. Assume also that

$$\liminf_{t\to\infty} \pi^t(z, B(z,r)) > 0 \quad \text{for any } r > 0. \tag{2.12}$$

Then

$$\operatorname*{w-lim}_{t\to\infty} P^t\nu = \mu_* \tag{2.13}$$

for any $\nu \in \mathcal{M}_1$ that is supported in $\mathcal{T}$.

If we replace condition (2.12) with the stronger condition

$$\inf_{x\in X}\liminf_{t\to\infty} \pi^t(x, (B(z,r)) > 0 \quad \text{for any } r > 0, \tag{2.14}$$

then $\mathbf{\Phi}$ is asymptotically stable.

3. Iterated Function Systems We are given a sequence of continuous transformations

$$S_i : X \to X \qquad \text{for } i = 1, \dots, N \tag{3.1}$$

and a probabilistic vector $(p_1(x), \dots, p_N(x))$, $x \in X$, i.e.,

$$p_i(x) \geqslant 0 \quad \text{and} \quad \sum_{i=1}^{N} p_i(x) = 1 \quad \text{for } x \in X.$$

We assume that p_i, $i = 1, \dots, N$, are also continuous functions. The pair of sequences $(S, p) = (S_1, \dots, S_N; p_1, \dots, p_N)$ is called an **Iterated Function System (IFS)**.

Now we present an imprecise description of the process considered in this section. Choose $x_0 \in X$. If an initial point x_0 is chosen, we randomly select an integer from the set $\{1, \ldots, N\}$ in such a way that the probability of choosing k is $p_k(x_0)$, $k = 1, \ldots, N$. When the number k_0 is drawn we define $x_1 = S_{k_0}(x_0)$. Having x_1, we select k_1 according to the distribution $p_1(x_1), \ldots, p_N(x_1)$ and we define $x_2 = S_{k_1}(x_1)$ and so on. Denoting by μ_n, $n = 0, 1, \ldots$, the distribution of x_n, i.e., $\mu_n(A) = \text{Prob}(x_n \in A)$ for $A \in \mathcal{B}(X)$, we define P as the transition operator such that $\mu_n = P\mu_{n-1}$ for $n \in \mathbb{N}$.

The above procedure can be formalized and we may easily show that the transition operator for the given **IFS**is of the form:

$$\pi(x, A) = \sum_{i=1}^{N} p_i(x)\mathbf{1}_A(S_i(x)) \qquad \text{for any } x \in X \text{ and } A \in \mathcal{B}(X). \tag{3.2}$$

Therefore,

$$P\mu(A) = \sum_{i=1}^{N} \int_{S_i^{-1}(A)} p_i(x)\mu(dx) \qquad \text{for } A \in \mathcal{B}(X) \tag{3.3}$$

and

$$Uf(x) = \sum_{i=1}^{N} p_i(x) f(S_i(x)) \qquad \text{for } x \in X. \tag{3.4}$$

Since $Uf \in C(X)$ for $f \in C(X)$, the Markov chain corresponding to the Iterated Function System satisfies the Feller property.

Let the iterated function system $(S, p)_N$ satisfy

$$\sum_{i=1}^{N} |p_i(x) - p_i(y)| \leqslant \omega(\rho(x, y)) \quad \text{for } x, y \in X, \tag{3.5}$$

and

$$\sum_{i=1}^{N} p_i(x)\rho(S_i(x), S_i(y)) \leqslant r\rho(x, y) \quad \text{for } x, y \in X \text{ with } r < 1. \tag{3.6}$$

We will assume that $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, nondecreasing and concave. Moreover $\omega(0) = 0$. If we assume also that

$$\int_0^{\varepsilon} \frac{\omega(t)}{t} \mathrm{d}t < \infty \qquad \text{for some } \varepsilon > 0,$$

then we will say that it satisfies the **Dini** condition.

From the Dini condition it follows that for any $c < 1$

$$\lim_{x \to x_0} \sum_{n=1}^{\infty} \omega(c^n \rho(x, x_0)) = 0. \tag{3.7}$$

On the other hand, easy computation shows that

$$|U^n f(x) - U^n f(y)| \leqslant L \sum_{i=1}^{n-1} \omega(r^n \rho(x,y))$$

for $n \geqslant 1$ and any Lipschitz function f with the Lipschitz constant L, and consequently $\{U^n f : n \geqslant 1\}$ is equicontinuous, by (3.7).

The proof of stability for Markov chains corresponding to Iterated Function Systems satisfying (3.5), (3.6) and the Dini condition was given by Barnsley *et al.* in [1] (see also Lasota and Yorke in [14]) in the case when the state space X is finitely dimensional. Their proof may be extended to infinitely dimensional spaces if we know that the chain has an invariant measure. The existence of an invariant measure follows, however, from Theorem 2.1, since the sequence $\{U^n f : n \geqslant 1\}$ is equicontinuous (see also [17]). It is worth mentioning here that the Dini condition is essential for stability. We may formulate the following theorem.

Theorem 3.1 Let an iterated function system (S, p) satisfy (3.5), (3.6) and the Dini condition. If

$$\sum p_i(x) p_i(y) > 0 \qquad \text{for } x, y \in X,$$

where the summation is taken over all $i \in \{1, \dots, N\}$ such that $\rho(S_i(x), S_i(y)) \leqslant r\rho(x,y)$, then the Markov chain corresponding to (S, p) is asymptotically stable.

4. Stochastic partial differential equations

Using Theorem 2.3, we may establish the existence of an invariant measure and stability for the family defined by a stochastic evolution equation of the form

$$\mathrm{d}Z(t) = (AZ(t) + F(Z(t)))\,\mathrm{d}t + R\mathrm{d}W(t). \tag{4.1}$$

Here we assume that $\mathcal{X}$ is a real separable Hilbert space, A is the generator of a C_0-semigroup $S = (S(t))_{t \geqslant 0}$ acting on $\mathcal{X}$, F is a mapping (not necessarily continuous) from $D(F) \subset \mathcal{X}$ to $\mathcal{X}$, R is a bounded linear operator from another Hilbert space $\mathcal{H}$ to $\mathcal{X}$, and $W = (W(t))_{t \geqslant 0}$ is a cylindrical Wiener process on $\mathcal{H}$ defined over a certain filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geqslant 0}, \mathbb{P})$.

Let Z_0 be an $\mathcal{F}_0$-measurable random variable. By a solution of (4.1) starting from Z_0 we mean a solution to the stochastic integral equation of the following form (a so called *mild solution*)

$$Z(t) = S(t)Z_0 + \int_0^t S(t-s)F(Z(s))\mathrm{d}s + \int_0^t S(t-s)R\mathrm{d}W(s), \qquad t \geqslant 0,$$

see e.g. [3], where the stochastic integral appearing on the right hand side is understood in the sense of Itô. We suppose that for every $x \in \mathcal{X}$ there is a unique mild solution $Z^x = (Z^x_t)_{t\geqslant 0}$ of (4.1) starting from x, and that (4.1) defines in this way a Markov family. The corresponding semigroup is given by $U^t\psi(x) = \mathbb{E}\,\psi(Z^x(t))$, $t \geqslant 0$, $\psi \in B_b(\mathcal{X})$, $x \in \mathcal{X}$. We assume that it is Feller.

A function $\Phi\colon \mathcal{X} \mapsto [0, +\infty)$ will be called a **Lyapunov function**, if it is measurable and

$$\lim_{\|x\|_{\mathcal{X}}\to\infty} \Phi(x) = \infty.$$

We shall assume that the deterministic equation

$$\frac{\mathrm{d}Y(t)}{\mathrm{d}t} = AY(t) + F(Y(t)), \qquad Y(0) = x \tag{4.2}$$

defines a continuous semi-dynamical system, i.e. for each $x \in \mathcal{X}$ there exists a unique continuous solution to (4.2) that we denote by $Y^x = (Y^x(t), t \geqslant 0)$ and for a given t the mapping $x \mapsto Y^x(t)$ is measurable. Furthermore, we have $Y^{Y^x(t)}(s) = Y^x(t+s)$ for all $t, s \geqslant 0$ and $x \in \mathcal{X}$.

A set $\mathcal{K} \subset \mathcal{X}$ is called a **global attractor** for equation (4.2) if

- it is invariant under the semi-dynamical system generated by (4.2), i.e. $Y^x(t) \in \mathcal{K}$, $t \geqslant 0$ for any $x \in \mathcal{K}$,
- for any $\varepsilon, R > 0$ there exists T such that $Y^x(t) \in \mathcal{K} + \varepsilon B(0,1)$ for $t \geqslant T$ and $\|x\|_{\mathcal{X}} \leqslant R$.

The family $(Z^x(t))_{t\geqslant 0}$, $x \in \mathcal{X}$, is **stochastically stable** if for every ε, R, $t > 0$

$$\inf_{x\in B(0,R)} \mathbb{P}\left(\|Z^x(t) - Y^x(t)\|_{\mathcal{X}} < \varepsilon\right) > 0. \tag{4.3}$$

We derive from Theorem 2.3 the following result concerning the stability of Z.

Theorem 4.1 **(ref. [18])** Assume that:

- the semi-dynamical system $(Y^x(t), t \geqslant 0)$ defined by (4.2) possesses a global attractor $\mathcal{K}$
- there exists a certain Lyapunov function Φ such that

$$\sup_{t\geqslant 0} \mathbb{E}\,\Phi(Z^x(t)) < \infty \qquad \text{for any } x \in \mathcal{X},$$

- the family $(Z^x(t))_{t\geqslant 0}$, $x \in \mathcal{X}$, is stochastically stable and its semigroup $\{U^t\phi : t \geqslant 0\}$ is equicontinuous for any Lipschitz function ϕ.

Then, the corresponding transition semigroup $(P_t)_{t\geqslant 0}$ admits an invariant measure. Moreover, if the attractor $\mathcal{K}$ is a singleton, then $(P_t)_{t\geqslant 0}$ is asymptotically stable.

References

[1] Barnsley M.F., Demko S.G., Elton J.H. and Geronimo J.S. *Invariant measures arising from iterated function systems with place dependent probabilities* Ann. Inst. Henri Poincaré **24**(3) (1988), 367–394. MR 0971099; Zbl 0653.60057

[2] Billingsley, P., *Convergence of Probability Measures*, Wiley, New York, 1968. xii+253 pp. MR 0233396; Zbl 0172.21201

[3] Da Prato, G. and Zabczyk, J., *Stochastic Equations in Infinite Dimensions. Encyclopedia of Mathematics and Its Applications.* vol. 44 Cambridge University Press, Cambridge 1992. xviii+454 pp. ISBN: 0-521-38529-6 MR 1207136 ; Zbl 0761.60052

[4] Doeblin, W., *Éléments d'une théorie générale des chaines simples constantes de Markov*, Ann. Ecole Norm. **57** (1940), 61–111. MR 0004409; Zbl 0303.9169

[5] Es-Sarhir, A. and von Renesse, M.-K., *Ergodicity of stochastic curve shortening flow in the plane*, SIAM J. Math. Anal. **44 (1)** (2012), 224–244. MR 2888287 ; Zbl 1251.47041; doi: 10.1137/100798235

[6] Ethier, S. and Kurtz, T., *Markov processes: Characterization and Convergence*, Wiley, New York, 1968. MR 0838085; Zbl 1089.60005

[7] Hairer, M. and Mattingly, J., *Ergodicity of the 2D Navier-Stokes equations with degenerate stochastic forcing*, Ann. of Math. **164** (2006), 993–1032. MR 2259251; Zbl 1130.37038; doi: 10.4007/annals.2006.164.993

[8] Jaroszewska, J., *On asymptotic equicontinuity of Markov transition functions*, Statistics and Probability Letters **83** (2013), 943-951. MR 3040327; Zbl 1268.60102; doi: 10.1016/j.spl.2012.10.033

[9] Kapica, R., Szarek T. and Ślęczka, M., *On a unique ergodicity of some Markov processes*, Potential Anal. **36** (2012), 589-606. MR 2904635; Zbl 1244.60074; doi: 10.1007/s11118-011-9242-0

[10] Komorowski, T., Peszat, S. and Szarek, T., *On ergodicity of some Markov processes*, Ann. Probab. **38 (4)** (2010), 1401-1443. MR 2663632; Zbl 1214.60035; doi: 10.1214/09-AOP513

[11] Lasota, A. and Mackey, M., *Probabilistic Properties of Deterministic Systems*, Cambridge University Press, Cambridge 1985. MR 1244104; Zbl 0784.58005

[12] Lasota, A. and Szarek, T., *Lower bound technique in the theory of a stochastric differential equation*, J. Differential Equations **231** (2006), 513–533. MR 2287895; Zbl 05115329

[13] Lasota, A. and Yorke, J. A., *On the existence of invariant measures for piecewise monotonic transformations*, Trans. Amer. Math. Soc. **186** (1973), 481–488. MR 0335758; Zbl 0298.28015; doi: 10.1090/S0002-9947-1973-0335758-1

[14] Lasota, A. and Yorke, J. A., *Lower bound technique for Markov operators and iterated function systems*, Random Comput. Dynam. **2** (1994), 41–77. MR 1265226; Zbl 0804.47033; doi: 10.1016/j.jde.2006.04.018

[15] Meyn, S.P. and Tweedie, R. L., *Markov Chains and Stochastic Stability*, Springer-Verlag, Berlin, Heidelberg, New York, 1993. ; MR 1287609; Zbl 1165.60001; doi: 10.1017/CBO9780511626630

[16] Szarek, T., *Feller processes on nonlocally spaces*, Ann. Probab. **34** (2006), 1849–1863. MR 2271485; Zbl 1108.60064; doi: 10.1214/009117906000000313

[17] Szarek, T., *Invariant measures for nonexpansive Markov operators on Polish spaces*, Dissert. Math. **415**, (2003), p. 62. MR 1997024; Zbl 1051.37005; doi: 10.4064/dm415-0-1

[18] Szarek, T., Ślęczka, M. and Urbański, M. *On stability of velocity vectors for some passive tracer models*, Bull. London Math. Soc. **42 (5)** (2010), 923-936. MR 2728696; Zbl 1204.60070; doi: 10.1112/blms/bdq055

[19] Szarek, T. and Worm, D.T.H. *Ergodic measures of Markov semigroups with the e-property*, Ergodic Theory Dynamical Systems **32 (3)** (2012), 1117-1135. MR 2995659; Zbl 1261.37007; doi: 10.1017/S0143385711000022

[20] Xie, B. *Uniqueness of invariant measures of infinite dimensional stochastic differential equations driven by Lévy noises*, Potential Anal. **36 (1)** (2012), 35–66. MR 2886453; Zbl 1260.60126; doi: 10.1007/s11118-011-9220-6

[21] Wang, J. *Regularity of semigroups generated by Lévy type operators via coupling*, Stochastic Process. Appl. **120 (9)** (2010), 1680–1700. MR 2673970; Zbl 1204.60071; doi: 10.1016/j.spa.2010.04.007

[22] Zaharopol, R., *Invariant Probabilities of Markov-Feller Operators and their Supports*, Birkhäuser, Basel 2005. MR 2128542; Zbl 1072.37007

Metoda miary dolnej i jej zastosowanie w teorii iterowanych systemów funkcyjnych i stochastycznych równań różniczkowych

Tomasz Szarek

Streszczenie W pracy formułujemy kryteria dla istnienia miary niezmienniczej dla łańcuchów i procesów Markowa. Następnie pokazujemy ich użyteczność w teorii iterowanych układów funkcyjnych i stochastycznych równań różniczkowych.

2010 Klasyfikacja tematyczna AMS (2010): 60J25, 60G51.

Słowa kluczowe: Ergodyczność, rodziny Markowa, miara niezmiennicza, iterowane systemy funkcji, stochastyczne równania różniczkowe.

1. Wprowadzenie Celem pracy jest przybliżenie techniki miar dolnych w teorii ergodycznej procesów Markowa. Technika ta używana była przez Doeblina (patrz [4]) w celu pokazania własności mieszania łańcucha Markowa, którego prawdopodobieństwa przejścia są jednostajnie ograniczone z dołu. Jest to inna technika niż ta oparta na analizie operatora dualnego do prawdopodobieństw przejścia, stosowana przez Lasotę i Yorke'a (patrz np. citelasota-mackey). Przykładowo, w [13] pokazali, że istnienie dolnego ograniczenia dla iteracji operatora Frobeniusa–Perrona, co odpowiada przedziałowej monotonicznej transformacji przedziału jednostkowego, implikuje istnienie rozkladu stacjonarnego dla deterministycznego łańcucha Markowa opisującego iteracje przekształcenia. W rzeczywistości miara niezmiennicza jest jedyna w klasie miar absolutnie ciągłych względem jednowymiarowej miary Lebesque'a i statystycznie stabilna, tzn. rozkład jednowymiarowy, jeśli rozkład początkowy jest absolutnie ciągły, dąży do miary niezmienniczej w metryce całkowitej wariacji miary. Technika ta została rozszerzona na ogólniejsze łańcuchy Markowa, włączając te związane z systemami funkcji iterowanych (patrz np. [14]). Jednakże, większość rezultatów została sformułowana dla łańcuchów Markowa o wartościach w pewnych skończenie wymiarowych przestrzeniach (więcej szczegółów można znaleźć w artykule przeglądowym [22]). To założenie jest bardzo zawężające ponieważ wiele zastosowań, które chcemy objąć, jest z wykorzystaniem przestrzeni nieskończenie wymiarowych (chocby pewne przestrzenie funkcyjne w przypadku stochastycznych równań różniczkowych cząstkowych).

W literaturze można znaleźć dużo procesów o tej własności intensywnie badanych w ostatnim okresie (patrz np. [7–9, 18, 20, 21]). Dla e–procesów możemy sformułować kryteria istnienia miar niezmienniczych i ich stabilności. Takie kryteria zaprezentowano w rozdziale **??**. Section 3 przybliża pewne zastosowania uzyskanych rezultatów w teorii systemów funkcji iterowanych (**IFS's**) oraz stochastycznych równań różnicznkowych o pochodnych cząstkowych (**SPDE's**).

2. Operatory Markowa i półgrupa operatoró markowskich

3. System Iterated Function Systems Dany jest ciąg przekształceń ciągłych (3.1) oraz wektor probabilistyczny $(p_1(x), \ldots, p_N(x))$ zależny od $x \in X$. Zakładamy, że p_i, $i = 1, \ldots, N$ są ciągłe na X. Parę ciągów funkcji $(S, p) = (S_1, \ldots, S_N; p_1, \ldots, p_N)$ nazywamy iteracyjnym systemem funkcji (*IFS*).

Rozważamy procedurę którą nieprecyzyjnie można opisać następująco. Wybieramy $x_0 \in X$ oraz liczbę całkowitą z $\{1, \ldots, N\}$ zgodnie z rozkładem $p_k(x_0)$, $k = 1, \ldots, N$, a następnie definiujemy $x_1 = S_{k_0}(x_0)$. Wykorzystując x_1 wynieramy k_1 zgodnie z rozkładem $\{p_i(x_1)\}_{i=1}^{N}$ o definiujemy $x_2 = S_{k_1}(x_1)$, itd. Niech μ_n, $n = 0, 1, \ldots$ będzie rozkładem x_n, tzn. $\mu_n(A) = \text{Prob}(x_n \in A)$ gdy $A \in \mathcal{B}(X)$. Definiujemy operator przejścia P formułą $\mu_n = P\mu_{n-1}$ dla $n \in \mathbb{N}$.

Po sformalizowaniu tej procedury można udowodnić, że operator przejścia zadany przez *IFS* ma postać (3.2) i mamy (3.3) oraz (3.4). Ponieważ $Uf \in C(X)$ dla $f \in C(X)$, stąd łańcuch Markowa odpowiadający *IFS* ma własność Fellera.

4. Stochastyczne równania różniczkowe o pochodnych cząstkowych Wykorzystując Twierdzenie 2.3 możemy uzasadnić istnienie miary niezmienniczej oraz stabilność dla rodziny definiowanej ewolucja stochastyczną (4.1).

Tomasz Szarek is a Professor in the Department of Mathematics at the University of Gdańsk. His PhD was conducted in the Institute of Mathematics the Polish Academy of Sciences in Warsaw under the supervision of Professor A. Lasota. He received his Habilitation from the Jagiellonian University in Cracow. Recently he earned his second PhD, in analytic philosophy, at the University of Silesia.

Tomasz Szarek
University of Gdańsk
Institute of Mathematics
Wita Stwosza 57, 80-952 Gdańsk, Poland
E-mail: szarek@intertele.pl
Communicated by: Łukasz Stettner

(Received: 11th of October 2013)