Non-continuous interval maps and rotation theory in spiking neuron models

Justyna Signerska-Rynkowska

Mathematical Neuroscience Lab, CIRB — Collège de France; INRIA-Rocquencourt, EPI MYCENAE

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1 Neuron models

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3 Rotation theory
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What characterizes neurons’ activity? Neurons are electrically excitable cells that communicate through the emission of action potentials (spikes): stereotyped membrane potential electrical impulses. They encode information in the way spikes are emitted through:

- the answer to specific simple stimuli (excitability properties)
- and the spike pattern fired,
- often related to properties of the interspike behavior (e.g.: sub-threshold oscillations).
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- often related to properties of the interspike behavior (e.g.: sub-threshold oscillations).
Different spike patterns

Markram et al 2004
Phenomenological Neuron Models

Aimed to reproduce the typical behaviors of nerve cells in response to different stimuli:

- Excitability properties of neurons
- Frequency preference property
- subthreshold oscillations,
- Spike patterns fired
The main excitability properties can be linked with bifurcations of dynamical systems for

- **Continuous dynamical systems**: detailed neuron models and their reductions (Rinzel, Ermentrout, Guckenheimer, . . . ).
- **Discrete dynamical systems**: map-based models (Caselles, Rulkov, . . . )

**Hybrid dynamical systems**

Integrate-and-fire neuron models combine:

- A **continuous** dynamical system (ordinary differential equations) accounting for input integration
- A **discrete** dynamical system (map iteration) accounting for spike emission.
Classical Integrate-and-Fire Neurons

$$\frac{dv}{dt} = -v + I$$
$$v = \theta \Rightarrow \text{Spike!}$$

Louis Lapicque, 1907


$$\begin{cases} \dot{v} = v^2 - w + I \\ \dot{w} = a(bv - w) \end{cases} \quad \begin{cases} \dot{v} = e^v - v - w + I \\ \dot{w} = a(bv - w) \end{cases}$$
Classical Integrate-and-Fire Neurons

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\frac{dv}{dt} = -v + I \\
v = \theta \Rightarrow \text{Spike!}
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\dot{v} = -v - w + I \\
\dot{w} = a(bv - w)
\]

Wehmeier et al, 1989

Izhikevich (2003)  
Brette & Gerstner (2005)
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\[
\begin{align*}
\dot{v} &= v^2 + I \\
\dot{w} &= e^v - v + I
\end{align*}
\]

e.g. Ermentrout Kopell, 1982, Fourcaud-Trocme et al 2003


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One-dimensional models
\[
\dot{x} = F(t, x) \quad F : \mathbb{R}^2 \rightarrow \mathbb{R}
\]

\[
\lim_{s \to t^+} x(s) = 0, \quad \text{if} \quad x(t) = 1
\]

**Definition [Firing map]**

\[
\Phi : \mathbb{R} \rightarrow \mathbb{R}, \quad \Phi(t) = \min\{s > t : x(s; t, 0) = 1\}
\]

\[
D_\Phi = \{t \in \mathbb{R} : \exists s > t \ x(s; t, 0) = 1\}
\]

\[
t_n = \Phi^n(t) = \min\{s > \Phi^{n-1}(t) : x(s; \Phi^{n-1}(t), 0) = 1\}
\]
Perfect Integrator Model

\[ \dot{x} = f(t) \]  

(PI)

Leaky Integrate-and-Fire

\[ \dot{x} = -\sigma x + f(t) \]  

(LIF)

Non-linear models

\[ \dot{x} = F(t, x) \]

- Mathematical analysis of one-dimensional IF models was performed e.g. in [R.Brette, 2004], [H.Carrillo, F.A.Ongay, 2001], [T.Gedeon, M.Holzer, 2004], [W. Marzantowicz, J.S., 2011], [W. Marzantowicz, J.S., 2015] and [P. Kasprzak, A. Nawrocki, J. S., 2015] (with focus on periodic and almost-periodic input functions)
Bidimensional integrate-and-fire models
Bidimensional integrate-and-fire models:

\[ \dot{v} = F(v) - w + I \]  \hspace{1cm} (1)
\[ \dot{w} = a(bv - w) \]  \hspace{1cm} (2)

A spike is emitted at time \( t^* \) such that

\[ \lim_{t \to t^*^-} v(t) = \infty \]

At the moment of the spike we reset:

\[ v(t^*) \rightarrow v_R, \quad w(t^*) \rightarrow \gamma w(t^*-) + d \]

The adaptation map: \( \Phi(w_0) = \gamma w(t^*-) + d \), \((v_R, w_0)\)-the initial condition of the solution \((v(t), w(t))\) which spikes at \( t^* \).

Examples include adaptive exponential model \((F(v) = e^v - v)\), quadratic adaptive model \((F(v) = v^2)\) and quartic model \((F(v) = v^4 + 2av)\).
Bidimensional integrate-and-fire models:

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\dot{v} &= F(v) - w + I \\
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\end{align*} \tag{1} \tag{2} \]

A spike is emitted at time \( t^* \) such that

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At the moment of the spike we reset:

\[ v(t^* +) \rightarrow v_R, \quad w(t^* +) \rightarrow \gamma w(t^* -) + d \]

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Examples include *adaptive exponential model* \((F(v) = e^v - v)\), *quadratic adaptive model* \((F(v) = v^2)\) and *quartic model* \((F(v) = v^4 + 2av)\)
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\[ \frac{dw}{dt} = a(bv - w) \]  

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At the moment of the spike we reset:

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Behaviors of the Quartic Model
The model can display complex dynamics including **Mixed-Mode Oscillations** and **Mixed-Mode Bursting Oscillations** (MM(B)O) that are sequences of spikes interspersed by small subthreshold oscillations.

MM(B)Os so far have been investigated in 3D and higher dimensional systems ([M. Desroches et al., 2012], [M. Krupa et al., 2012], [T. Vo et al., 2012]).

In such hybrid models they have never been observed before.

From the neuroscience point of view, they have been evidenced in Hodgkin-Huxley model ([J. Rubin, M. Wechselberger, 2007], [J. Rubin, M. Wechselberger, 2008]) and in the coupled FitzHugh-Nagumo systems ([N. Berglund, D. Landon, 2012], [M. Desroches et al., 2008]).
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Another (classical) example are chemical reactions:

Bromide ion electrode potential in the Belousov–Zhabotinsky reaction; figure from [J.L. Hudson et al., 1979]
Our aim was to show that they also occur in 2D integrate-and-fire models through the simple geometric mechanism.

[joint work with J. Touboul (Mathematical Neuroscience Lab and EPI MYCENAE) and A. Vidal (LaMME, Univ Evry and EPI MYCENAE)]
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Mixed-Mode Oscillations (MMOs, slow oscillations interspersed with spikes or bursts):
Bidimensional integrate-and-fire models were studied in [R. Brette, W. Gerstner, 2005], [E. Izhikevich, 2003], [N. Jimenez et al., 2013] and [J. Touboul, R. Brette, 2009].

We assume that:
- $F \in C^3(\mathbb{R})$ (at least)
- $F$ is strictly convex
- $\lim_{\nu \to -\infty} F'(\nu) < 0$
- there exist $\varepsilon > 0$ and $\delta > 0$ such that:
  $$\lim_{\nu \to \infty} \frac{F(\nu)}{\nu^{2+\varepsilon}} \geq \delta$$
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  $$\lim_{v \to \infty} \frac{F(v)}{v^{2+\varepsilon}} \geq \delta$$
The adaptation map

We define:

- $\mathcal{D}$ the set of $w$ s.t. the solution starting from $(v_r, w)$ spikes.
- $\Phi: \mathcal{D} \leftrightarrow \mathbb{R}$ the function such that $\Phi(w)$ is the after-spike adaptation value.
**Definition [Adaptation map]**

The adaptation map $\Phi$ associates to a value of the adaptation variable $w$ the value of the adaptation variable after reset:

$$\Phi(w) := \gamma W(t^*; v_r, w) + d,$$

where $(V(t; v_r, w), W(t; v_r, w))$ is the solution of the system (1)-(2) with initial condition $(v_r, w)$ at time $t$, and $t^*$ is the value at which $V(t; v_r, w)$ diverges.

Let $\mathcal{D} = \{w_1, w_2, \ldots\}$ be the set of intersections of the line $v = v_R$ with SMSFP. Then $\Phi : \mathbb{R} \setminus \mathcal{D} \rightarrow \mathbb{R}$ is well-defined.
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Let $D = \{w_1, w_2, \ldots\}$ be the set of intersections of the line $v = v_R$ with SMSFP. Then $\Phi : \mathbb{R} \setminus D \to \mathbb{R}$ is well-defined.
Neuron models

Geometric mechanism for MMO

Rotation theory

Conclusions

Phase Space

Adaptation Map

Generated signal
Remark

Suppose that \((v(t), w(t))\) is the spiking solution starting from \((v_r, w_0)\) and let \(\{t_n\}_{n>0}\) be the sequence of spike times for this solution. By \(\{w_n\}_{n>0}\) denote the values of the adaptation variable \(w\) at spikes, i.e.

\[
    w_n := w(t_n^+) = \gamma w(t_n^-) + d
\]

Then the adaptation map satisfies

\[
    \Phi(w_n) = w_{n+1}
\]

The spike train can be qualitatively described via iterations of \(\Phi\), with fixed points of \(\Phi\) corresponding to tonic, regular spiking and periodic orbits to bursts. Thus the study of the dynamics of \(\Phi\) allows to discriminate between different spiking patterns.
Remark

Suppose that \((v(t), w(t))\) is the spiking solution starting from \((v_r, w_0)\) and let \(\{t_n\}_{n>0}\) be the sequence of spike times for this solution. By \(\{w_n\}_{n>0}\) denote the values of the adaptation variable \(w\) at spikes, i.e.

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Neuron models

Geometric mechanism for MMO

Rotation theory

Conclusions

(figures from [J. Touboul, R. Brette, 2009])
Neuron models

Geometric mechanism for MMO

Rotation theory

Conclusions

- $(w_i)_{i=1}^p$ - intersections of the reset line $\{v = v_r\}$ with SMSFP
- $p_1$ - the index such that $(w_i)_{i \leq p_1}$ are below the $v$-nullclines and $(w_i)_{i > p_1}$ are above
- $(l_i)_{i=0}^{p+1}$ - intervals with endpoints $w_i$
- $\alpha, \beta$ - the value of $w$ after a spike for an initial condition on the upper and, respectively, lower branch of UMSFP
The adaptation map has the following properties:

1. **it is defined for all** \( w \in D = \mathbb{R} \setminus \{ w_i; i = 1 \cdots p \} \)
2. **its regular (at least \( C^1 \)) everywhere except the points** \( (w_i)_{i=1 \cdots p} \)
3. **at the boundaries of the definition domain** \( D, \{ w_i; i = 1 \cdots p \} \), the map has well-defined and distinct left and right limits:

\[
\begin{align*}
\lim_{w \to w_i^-} \Phi(w) &= \alpha, & \lim_{w \to w_i^+} \Phi(w) &= \beta, & i \leq p_1 \\
\lim_{w \to w_j^-} \Phi(w) &= \beta, & \lim_{w \to w_j^+} \Phi(w) &= \alpha, & j > p_1
\end{align*}
\]

4. **the derivative** \( \Phi'(w) \) **diverges at the discontinuity points:**

\[
\begin{align*}
\lim_{w \to w_i^\pm} \Phi'(w) &= \infty & i \leq p_1 \\
\lim_{w \to w_i^\pm} \Phi'(w) &= -\infty & i > p_1
\end{align*}
\]
for $w < \min\{\frac{d}{1-\gamma}, w_1, w^{**}\}$ we have $\Phi(w) \geq \gamma w + d > w$

$\Phi(w)$ is convex in the left-neighbourhood of $w_i$ (and concave in the right-neighbourhood)

$\Phi(w)$ has a horizontal plateau for $w \to \infty$ provided that

$$\lim_{v \to -\infty} F'(v) < -a(b + \sqrt{2})$$
The divergence of the derivative \( \lim_{w \to w_1} \Phi'(w) = \infty \) is due to the magnitudes of the eigenvalues \( \nu > 0 \) and \( \mu < 0 \) of the saddle fixed point: \( |\nu| - \mu > 0 \)
Assume that the line $v = v_r$ has two intersections with SMSFP: $w_1$ and $w_2$, with $w_1 < w_2$. We distinguish the following cases:

I. $\beta < w_1 < \alpha < w_2$

I’. $\beta < w_1 < w_* < w_2 < \alpha$

II. $\alpha < w_* < w_2$

II’. $w_* \leq \alpha < w_2$

III. $\Phi(\beta) \geq \beta$

III’. $\Phi(\beta) < \beta$

I.” $\beta < \alpha < w_1$

I.”’ $w_1 < \beta < \alpha$

IV.a $\Phi(\alpha) \leq \Phi(\beta)$

IV.b $\Phi(\alpha) > \Phi(\beta)$

V.a $w_1 < w_2 < \beta < \alpha$

V.b $w_1 < \beta < w_2 < \alpha$

V.c $w_1 < \beta < \alpha < w_2$
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Quartic model $F(\nu) = \nu^4 + 2c\nu$ with parameter values: $a = 0.1$, $b = 1$, $c = 0.1$, $l = -3(a/4)^{(4/3)}(2a - 1) + 0.1 \approx 0.1175$ and $\nu_r = 0.1158$
Geometric mechanism for MMO

Rotation theory

Conclusions

![Graph showing overlapping regions and fixed points](image)
Firstly, let us assume that

1. $\beta < w_1 < \alpha < w_2$

The above means that the identity line passes through the gap at $w_1$ and that in the interval $(-\infty, \alpha]$ (where the dynamics concentrates) there is only one discontinuity point $w_1$.

The analysis of $\Phi$ will be cut to the invariant interval $[x, \alpha]$, where $x = \beta$ when $\Phi(\beta) \geq \beta$ or $x = w_f$ when $\Phi(\beta) < \beta$ and $w_f < \beta$ is the greatest fixed point of $\Phi$ in $(-\infty, \beta)$.
Now let us add the following two assumptions about $\Phi$:

II. $\alpha < w_* < w_2$
III. $\Phi(\beta) \geq \beta$

**Proposition**

Under I. and II., whenever $\Phi : [x, \alpha] \setminus \{w_1\} \rightarrow [x, \alpha]$ has a periodic orbit (with period $q > 1$), this periodic orbit exhibits MMBO. However, this orbit does not need to be stable.
Assume I., II. and III.

We analyze $\Phi : [\beta, \alpha] \rightarrow [\beta, \alpha]$:

- $\Phi(w)$ is piecewise $C^1$ on $[\beta, \alpha]$ with a single jump discontinuity at $w = w_1 \in (\beta, \alpha)$.
- $\lim_{w \rightarrow w_1^-} \Phi'(w) = \lim_{w \rightarrow w_1^+} \Phi'(w) = \infty$
- $\lim_{w \rightarrow w_1^+} \Phi(w) = \beta$ and $\lim_{w \rightarrow w_1^-} \Phi(w) = \alpha$
According to [J.P. Keener, 1980] analysis of such maps can be performed separately for the following cases:

(i) non-overlapping case:
\[ \Phi(\alpha) < \Phi(\beta) \]

(ii) overlapping case:
\[ \Phi(\alpha) > \Phi(\beta) \]

(iii) \[ \Phi(\alpha) = \Phi(\beta) \]

with the help of the rotation number:

\[
\varrho(w) := \lim_{n \to \infty} \frac{\psi^n(w) - w}{n(\alpha - \beta)} = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \chi(w_{i+1}, \alpha)(\Phi^i(w))
\]
Non-overlapping case: I., II., III. and IV. $\Phi(\alpha) < \Phi(\beta)$
We consider the lift $\Psi : \mathbb{R} \to \mathbb{R}$ of $\Phi$ by identifying $\alpha$ with $\beta$ and requiring that $\Psi(w + k(\alpha - \beta)) = \Psi(w) + k(\alpha - \beta)$, for all $k \in \mathbb{Z}$ and $w \in \mathbb{R}$.

**Theorem (cf. [R.Brette, 2003] and [F.Rhodes, Ch.Thompson, 1986])**

*The rotation number*

$$\lim_{n \to \infty} \frac{\Psi^n(w) - w}{n(\alpha - \beta)} = \varrho$$

exists and does not depend on $w \in [\alpha, \beta]$.

Moreover, if $\varrho = \frac{p}{q} \in \mathbb{Q}$, then all orbits $\{\Phi^n(w)\}$, $w \in [\beta, \alpha]$, tend to a periodic orbit with the same period $q$ and if $\varrho \notin \mathbb{Q}$, then all orbits have the same limit set which is either the whole $[\beta, \alpha]$ or some Cantor subset of it (meaning, in particular there are no periodic orbits).
- $\varrho = 0 \mod 1 \implies$ tonic, regular spiking (for every initial condition $w_0 \in [\beta, \alpha] \setminus \{w_1\}$)
- $\varrho = p/q \in \mathbb{Q} \setminus \mathbb{Z} \implies$ MMBO (with periodicity of interspike-intervals and interspersing oscillations)
- $\varrho \in \mathbb{R} \setminus \mathbb{Q} \implies$ no periodic orbits and we observe chaos.
Neuron models

Geometric mechanism for MMO

Rotation theory

Conclusions
Proposition

Under I., II., III., IV.a., if $\Phi(\beta) > w_1$ and $\Phi(\alpha) < w_1$ then $\Phi$ has a periodic orbit of period two, which exhibits MMBO.
If $\Psi : \mathbb{R} \to \mathbb{R}$ is a non-decreasing map of degree-one (i.e. in our case $\Psi(w + (\alpha - \beta)) = \Psi(w) + (\alpha - \beta)$ for every $w \in \mathbb{R}$), then

$$R(\Psi) := \{(x, y) \in \mathbb{R}^2 : \Psi^-(x) \leq y \leq \Psi^+(x)\}. $$

**Definition [H-convergence]**

$\Psi_s \overset{H}{\to} \Psi_{s_0}$ as $s \to s_0$ iff $R(\Psi_s) \overset{R}{\to} (\Psi_{s_0})$ in the Hausdorff metric

We say that $(\Psi_s)$ is uniformly convergent to $\Psi_{s_0}$ at $x_0$ as $s \to s_0$ if for each $\varepsilon > 0$ there exist $\xi > 0$ and $\delta > 0$ such that for all $s$ and $x$ satisfying $|s - s_0| < \xi$ and $|x - x_0| < \delta$ we have $|\Psi_s(x) - \Psi_{s_0}(x_0)| < \varepsilon$. The $H$-convergence for non-decreasing degree-one circle maps can be characterised in a very convenient way:

**Proposition (cf. [F.Rhodes, Ch. Thompson, 1991])**

If $(\Psi_s)$ is a family of degree-one non-decreasing maps, then $\Psi_s \overset{H}{\to} \Psi_{s_0}$ as $s \to s_0$ if and only if $(\Psi_s)$ is uniformly convergent to $\Psi_0$ at each point of continuity of $\Psi_{s_0}$. 
If $\Psi : \mathbb{R} \to \mathbb{R}$ is a non-decreasing map of degree-one (i.e. in our case $\Psi(w + (\alpha - \beta)) = \Psi(w) + (\alpha - \beta)$ for every $w \in \mathbb{R}$), then

$$R(\Psi) := \{(x, y) \in \mathbb{R}^2 : \Psi^-(x) \leq y \leq \Psi^+(x)\}.$$

**Definition [H-convergence]**

$\Psi_s \xrightarrow{H} \Psi_{s_0}$ as $s \to s_0$ iff $R(\Psi_s) \xrightarrow{R} (\Psi_{s_0})$ in the Hausdorff metric.

We say that $(\Psi_s)$ is uniformly convergent to $\Psi_{s_0}$ at $x_0$ as $s \to s_0$ if for each $\varepsilon > 0$ there exist $\xi > 0$ and $\delta > 0$ such that for all $s$ and $x$ satisfying $|s - s_0| < \xi$ and $|x - x_0| < \delta$ we have $|\Psi_s(x) - \Psi_{s_0}(x_0)| < \varepsilon$. The $H$-convergence for non-decreasing degree-one circle maps can be characterised in a very convenient way:

**Proposition (cf. [F.Rhodes, Ch.Thompson, 1991])**

If $(\Psi_s)$ is a family of degree-one non-decreasing maps, then $\Psi_s \xrightarrow{H} \Psi_{s_0}$ as $s \to s_0$ if and only if $(\Psi_s)$ is uniformly convergent to $\Psi_0$ at each point of continuity of $\Psi_{s_0}$. 
If $\Psi : \mathbb{R} \to \mathbb{R}$ is a non-decreasing map of degree-one (i.e. in our case $\Psi(w + (\alpha - \beta)) = \Psi(w) + (\alpha - \beta)$ for every $w \in \mathbb{R}$), then

$$R(\Psi) := \{(x, y) \in \mathbb{R}^2 : \Psi^-(x) \leq y \leq \Psi^+(x)\}.$$ 

**Definition [H-convergence]**

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Theorem (cf. [R.Brette, 2003], [F.Rhodes, Ch. Thompson, 1991])

Suppose that $s \mapsto \Phi_s$, $s \in [c, d]$, is a family of adaptation maps with strictly increasing lifts $\Psi_s$ such that the mapping $(s, w) \mapsto \Psi_s(w)$ is increasing with respect to each variable and $s \mapsto \Psi_s$ is continuous with respect to the topology of $H$-convergence. Let $\varrho_s$ be the rotation number of $\Psi_s$. Then:

- $\rho : s \mapsto \varrho_s$ is continuous and non-decreasing;
- for all $p/q \in \mathbb{Q} \cap \text{Im}(\rho)$, $\rho^{-1}(p/q)$ is an interval containing more than one point, unless it is $\{c\}$ or $\{d\}$;
- $\rho$ reaches every irrational number at most once;
- $\rho$ takes irrational values on a Cantor-type subset of $[c, d]$, up to a countable number of points.
Proposition

Let $a$, $b$, $\nu_R$, $I$ and $\gamma$ be fixed and consider $d$ varying in some interval $d \in [\lambda_1, \lambda_2]$. Suppose that for this choice of parameter values $a$, $b$, $\nu_R$, $I$ and $\gamma$ the adaptation map $\Phi_d$ satisfies conditions I., II., III. and IV.a for any value of $d \in [\lambda_1, \lambda_2]$. Let $\varrho_d$ denote the unique rotation number obtained for the map $\Phi_d$ (considered on the "fundamental interval" $[\beta_d, \alpha_d]$). Then the mapping $\rho : d \mapsto \varrho_d$ is continuous.

If moreover, for every $d \in [\lambda_1, \lambda_2]$, the adaptation map $\Phi_d$ satisfies $\Phi_d(\beta_{\lambda_1}) > \Phi_d(\alpha_{\lambda_2})$, then the above mapping $\rho : d \mapsto \varrho_d$ behaves like a Devil’s staircase.
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Parameter values: $v_r = 0.1$, $\gamma = 0.05$
The rotation number $\varrho = \frac{p}{q} \in \mathbb{Q}$ characterises the signature of MM(B)O:

$$L_1^{s_1}, L_2^{s_2}, L_3^{s_3}, \ldots$$

where $L_i$ denotes the number of big oscillations (spikes) and $s_i$ is the number of following them small threshold oscillations.

For example, $\varrho = 1/3$ corresponds to the periodic signature $3^1$ and $\varrho = 3/5$ to the periodic signature $2^1, 1^1, 2^1$.
The rotation number $\varrho = p/q \in \mathbb{Q}$ characterises the signature of MM(B)O:

$$L_{s_1}^1, L_{s_2}^2, L_{s_3}^3, \ldots$$

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For example, $\varrho = 1/3$ corresponds to the periodic signature $3^1$ and $\varrho = 3/5$ to the periodic signature $2^1, 1^1, 2^1$. 
Overlapping case: I., II. and IV. b $\Phi(\beta) < \Phi(\alpha)$

The analysis of $\Phi : [\beta, \alpha] \to [\beta, \alpha]$ in the overlapping regime can be made via the results of [M. Misiurewicz, 1986] on *old heavy maps*.

Let $\Psi : \mathbb{R} \to \mathbb{R}$ denote the lift of $\Phi \upharpoonright [\beta, \alpha]$. The map $\Psi$ is a degree one map with only negative jumps. We can define the following:

**Definition [Rotation interval $\left[a(\Psi), b(\Psi)\right]$]**

\[
\begin{align*}
a(\Psi) & := \inf_{w \in \mathbb{R}} \liminf_{n \to \infty} \frac{\Psi^n(w) - w}{n(\alpha - \beta)}, \\
b(\Psi) & := \sup_{w \in \mathbb{R}} \limsup_{n \to \infty} \frac{\Psi^n(w) - w}{n(\alpha - \beta)}.
\end{align*}
\]

An old heavy map does not need to be monotonous in its intervals of continuity and therefore:

**Remark**

*If we assume IV.b, then we can skip the assumption II. since the induced lift $\Psi$ remains an old heavy map.*
Overlapping case: I., II. and IV. b $\Phi(\beta) < \Phi(\alpha)$

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**Definition [Rotation interval $[a(\Psi), b(\Psi)]$]**

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An old heavy map does not need to be monotonous in its intervals of continuity and therefore:

**Remark**

*If we assume IV.b, then we can skip the assumption II. since the induced lift $\Psi$ remains an old heavy map.*
Define:

\[ \psi_l(w) := \inf \{ \psi(z) : z \geq w \} \]
\[ \psi_r(w) := \sup \{ \psi(z) : z \leq w \} \]

The maps \( \psi_l(w) \) and \( \psi_r(w) \) are continuous and non-decreasing thus they admit unique rotation numbers.
The nontrivial rotation interval corresponds to complex dynamics (cf. [M. Misiurewicz, 1986]):

\( a \) if \( \Phi \) has a \( q \)-periodic point \( w \) with the rotation number \( \varrho(\psi, w) = p/q \), then \( a(\psi) \leq p/q \leq b(\psi) \);

\( b \) if \( a(\psi) < p/q < b(\psi) \), then \( \Phi \) has a periodic point \( w \) of period \( q \) and the rotation number \( \varrho(\psi, w) = p/q \)

The coexistence of periodic orbits with infinitely many different periods (non-triviality of the rotation interval) is also sometimes called chaos (see [J.P. Keener, 1980]).

**Remark**

If we additionally assume II., i.e. the monotonicity of \( \Phi \) in the continuity intervals \([\beta, w_1) \) and \((w_1, \alpha] \), then every such a periodic orbit exhibits MMBO (with both one and no small oscillations between consecutive spikes).
Theorem (cf. [M. Misiurewicz, 1986])

Suppose that the adaptation map $\Phi : [\beta, \alpha]$ is in the overlapping case and that for some $\varrho_1$ and $\varrho_2$ we have $a(\Psi) \leq \varrho_1 \leq \varrho_2 \leq b(\Psi)$. Then there exists $w_0$ such that

$$
\lim \inf_{n \to \infty} \frac{\Psi^n(w_0) - w_0}{n(\alpha - \beta)} := \varrho_1
$$

$$
\lim \sup_{n \to \infty} \frac{\Psi^n(w_0) - w_0}{n(\alpha - \beta)} := \varrho_2
$$

Proposition

Choose the fixed parameters $\nu_R$, $a$, $b$, $\gamma$ and $l$ and the parameter $d \in [\lambda_1, \lambda_2]$ such that for each $d \in [\lambda_1, \lambda_2]$ the map $\Phi_d$ is in the overlapping case, i.e. satisfies I., III. and IV.b. Then the maps $d \mapsto a(\Psi_d)$ and $d \mapsto b(\Psi_d)$, assigning to $d$ the endpoints of the rotation interval of $\Phi_d$, are continuous.
Moreover, usually the maps $d \mapsto a(\Psi_d)$ and $d \mapsto b(\Psi_d)$ also behave as Devil’s staircase:

Parameter values: $\nu_r = 0.1$, $\gamma = 0.05$
Theorem (Chaos)

Suppose that $\Phi$ satisfies I., II., III and IV. b (an overlapping case with additional monotonicity condition II.). Further assume also that $\Phi(\alpha) < w_1$ and that $\Phi$ has at least two periodic orbits, one with period $q_1$ and the other with period $q_2 \neq q_1$ and that exactly one point of each of these periodic orbits is greater than $w_1$. Then the mapping $w \mapsto \Phi(w)$ is a shift on a sequence space.

Theorem (Condition for orbits of all periods.)

Existence of a fixed point $w_f \in (\beta, w_1)$ and a periodic orbit with period $q > 1$ implies existence of periodic orbits with arbitrary periods $\tilde{q} > q$ and with MMBO. The same holds if $w_f \in (w_1, \alpha)$ provided that the $q$-periodic orbit is not of the type $q/q$ (i.e. it admits points to the left and to the right of $w_1$).

In particular, whenever there is a fixed point $w_f \in (\beta, \alpha)$ and a periodic orbit of the type $1/2$, then there are periodic orbits of all periods, exhibiting MMBO.
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Case of both positive and negative jumps: \{ I., II. and III.’ \} or \{ I., II.’ and III.’ \}

1. \( \beta < w_1 < \alpha < w_2 \)
2. \( \alpha < w_* < w_2 \)
3. \( w_* \leq \alpha < w_2 \)
4. \( \Phi(\beta) < \beta \)

Upper estimate of the rotation interval:

\[ [\varrho(\Psi_l), \varrho(\Psi_r)] \supset [a(\Psi), b(\Psi)] \]
Proposition

Under the assumptions \{ I., II. and III.’ \} or \{ I., II.’ and III.’ \}, if \( \Phi(\alpha) > w_1 \) and there are no fixed points in \((w_1, \alpha)\), then \( \Phi \) has an unstable periodic orbit of period 2. This orbit exhibits MMBO but it is unstable.
No discontinuity points in the invariant interval: $I$."

Under the following condition

$$\beta < \alpha < w_1$$

there are no discontinuity points $w_1$ or $w_2$ in the interval $(-\infty, \alpha)$. This is the easiest situation:

- since $\Phi(w) > w$ for $w < \min\{\frac{d}{1-\alpha}, w_1, w^{**}\}$ there must be a fixed point in $(-\infty, \alpha)$ and every point $w$ tends under $\Phi$ to one of the fixed points. Thus here we observe for every initial condition tonic, regular spiking (in particular, we have no MMO and MMBO) and the dynamics is very simple.
No discontinuity points in the invariant interval and the identity line passes below the gap at $w_1$: $I$."

$I.\" w_1 < \beta < \alpha$

$V.a \ w_1 < w_2 < \beta < \alpha$

**Theorem**

*Suppose that $\lim_{w \to \infty} \Phi(w) > w_2$. Then every point $w > w_1$ is forward asymptotic either to the fixed point $w_{f,1}$ or to a period two orbit. Under these assumptions no point $w \in \mathbb{R}$ exhibits MMO or MMBO.*
No discontinuity points in the invariant interval and the identity line passes below the gap at $w_1$: l.""

\[ l."\" w_1 < \beta < \alpha \]

\[ \text{V.b } w_1 < \beta < w_2 < \alpha \]

**Theorem**

*If $\Phi(w^*) < w_2$, then for every $w \in (w_1, w_2)$ we have $\omega(w) \subset \bar{P}$, where $\bar{P}$ denotes the closure of the set of periodic points of $\Phi : [w_1, w_2] \rightarrow [w_1, w_2]$. Particularly, if the set $P$ is finite, then every $w$ tends to some periodic orbit (or fixed point) (with no MM(B)O).*
No discontinuity points in the invariant interval and the identity line passes below the gap at $w_1$: I.’’’

I.’’’ $w_1 < \beta < \alpha$

V.c $w_1 < \beta < \alpha < w_2$

**Theorem**

Suppose that $\beta < w^* < \alpha$. If

$$\min \Phi^{-2}(w^*) < \Phi^2(w^*) < \min \Phi^{-1}(w^*) < w^* < \Phi(w^*)$$

and $\Phi(w) > w$ for $w \in (\beta, w^*)$, then $\Phi : [\beta, \alpha] \rightarrow [\beta, \alpha]$ has an orbit of period 3. Consequently, $\Phi$ has cycles of any period. However, these periodic orbits do not present MMBO.
Conclusions:

- We are able to predict the output properties using geometrical analysis.

- In the overlapping and non-overlapping cases existing mathematical tools of rotation theory provide complete description of the dynamics of $\Phi$.

- In the remaining cases (e.g. of both positive and negative jumps) one can obtain weaker results on the dynamics of $\Phi$; in particular the rotation interval computed via the enveloping maps $\Psi_i$ and $\Psi_r$ gives the upper-estimate for the possible types $p/q$ of periodic orbits.
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Generated signal
Perspectives:

- For multiple discontinuity points the dynamics is even more complex and harder to be completely classified. However, some rigorous results can be obtained via the theory of piece-wise continuous piece-wise monotone maps.

- Consider forcing of the IF system through variable $I$. A simple starting point is a square signal for $I(t)$: the performed analysis can be generalized using a stroboscopic map.

- Tackle the problematic of 3D vector field appearing with two recovery variables. In this case we have $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$. The general mechanism for generating MMBO is the same, yet leading to richer behaviors due to the geometric structure of the flow.
Thank you!
References


