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Weighted difference schemes for systems of quasilinear first order partial functional differential equations

Abstract The paper deals with initial boundary value problems of the Dirichlet type for system of quasilinear functional differential equations. We investigate weighted difference methods for these problems. A complete convergence analysis of the considered difference methods is given. Nonlinear estimates of the Perron type with respect to functional variables for given functions are assumed. The proof of the stability of difference problems is based on a comparison technique. The results obtained here can be applied to differential integral problems and differential equations with deviated variables. Numerical examples are presented.

2010 Mathematics Subject Classification: 35R10; 65M12; 65M15.

Key words and phrases: initial boundary value problems, difference methods, stability and convergence, interpolating operators, error estimates, comparison methods.

1. Introduction We are interested in numerical approximation of classical solutions to systems of quasilinear functional differential equations with initial boundary conditions. Difference schemes for first order partial functional differential equations are obtained by replacing partial derivatives with difference operators. Moreover, because differential equations contain functional variables which are elements of the class of continuous functions, some interpolating operators are needed. This leads to functional difference problems of Volterra type which satisfy consistency conditions on classical solutions of original problems.

The papers [17, 30] initiated the theory of difference methods for initial and initial boundary value problems for nonlinear functional differential equations of Hamilton Jacobi type. It is not our aim here to give a full review of papers concerning explicit difference methods for quasilinear functional differential equations. We shall mention only those which contain such reviews. They are [7, 9, 28, 32] and the monograph [16].

In recent years, a number of papers concerning implicit difference methods for functional partial differential equations have been published. Difference approximations of classical solutions to initial problems on the Haar pyramid

* This publication is co-financed by the European Union as part of the European Social Fund within the project Center for Applications of Mathematics.

and initial boundary value problems were investigated in [19, 20]. Implicit difference methods for parabolic equations with initial boundary conditions of the Dirichlet type were considered in [10, 21].

In the present paper we consider a difference method obtained in the following way. The partial derivatives with respect to spatial variable in functional differential equations are replaced by a suitable weighted difference operators. It means that with an appropriate value of weight we obtain explicit, implicit or strong implicit difference method. The papers [33, 34] consider weighted difference schemes for hyperbolic nonlinear partial functional differential equations. Results obtained in this paper and in [33, 34] are motivated by papers [24]- [26] where we can find an analysis of numerical methods with weight for nonlinear parabolic problems.

The authors of the papers [2]- [8], [15, 18, 20, 32, 33] have assumed that given functions satisfy the Lipschitz condition or nonlinear estimates of Perron type with respect to functional variables and these conditions are global. Assumptions which were adopted in this paper are more general. It is clear that there are differential equations with deviated variables and differential integral equations such that local estimates of the Perron type hold and global inequalities are not satisfied. In the paper we give suitable comments.

Theory of difference methods for functional differential equations with local estimates of the Perron type for given functions with respect to functional variable was initiated by the authors of the papers [11, 22, 29, 31]. The papers [11, 31] deal with initial problems for Hamilton Jacobi functional differential equations. Initial boundary value problems for nonlinear parabolic equations were investigated in [22, 29].

We formulate our functional differential problem. For any metric spaces X and Y we denote by $C(X, Y)$ the class of all continuous functions from X to Y . We will use vectorial inequalities with the understanding that the same inequalities hold between their corresponding components.

We consider the sets

$$\begin{aligned} E &= [0, a] \times [-b, b]^n, & E_0 &= [-b_0, 0] \times [-b, b]^n, \\ \partial_0 E &= [0, a] \times \left([-b, b]^n \setminus (-b, b)^n \right) \end{aligned}$$

where $a > 0$, $b_0 \in \mathbb{R}_+ = [0, \infty)$, $b = (b_1, \dots, b_n) \in \mathbb{R}^n$ and $b_i > 0$ for $1 \leq i \leq n$. By $[\dots]^n$ we define n -dimensional intervals. For $(t, x) \in E$ we define

$$D[t, x] = \left\{ (\tau, s) \in \mathbb{R}^{1+n} : \tau \leq 0, (t + \tau, x + s) \in E_0 \cup E \right\}.$$

Note that $D[t, x] = [-b_0 - t, 0] \times [-b - x, b - x]^n$. For a function $z : E_0 \cup E \rightarrow \mathbb{R}^k$ and for a point $(t, x) \in E$ we define a function $z_{(t,x)} : D[t, x] \rightarrow \mathbb{R}^k$ by

$$z_{(t,x)}(\tau, s) = z(t + \tau, x + s), \quad (\tau, s) \in D[t, x].$$

Then $z_{(t,x)}$ is the restriction of z to the set $(E_0 \cup E) \cap ([-b_0, t] \times \mathbb{R}^n)$ and this restriction is shifted to the set $D[t, x]$. Write $B = [-b_0 - a, 0] \times [-2b, 2b]^n$, then $D[t, x] \subseteq B$ for $(t, x) \in E$.

We use the notation $\mathbb{R}^{k \times n}$ for all $k \times n$ real matrices. Suppose that

$$f : E \times C(B, \mathbb{R}^k) \rightarrow \mathbb{R}^{k \times n}, \quad f = [f_{ij}]_{i=1, \dots, k, j=1, \dots, n},$$

$$g : E \times C(B, \mathbb{R}^k) \rightarrow \mathbb{R}^k, \quad g = (g_1, \dots, g_k),$$

$$\varphi : E_0 \cup \partial_0 E \rightarrow \mathbb{R}^k, \quad \varphi = (\varphi_1, \dots, \varphi_k)$$

are given functions. We consider the system of quasilinear differential functional equations

$$\partial_t z_i(t, x) = \sum_{j=1}^n f_{ij} \left(t, x, z_{(t,x)} \right) \partial_{x_j} z_i(t, x) + g_i \left(t, x, z_{(t,x)} \right), \quad 1 \leq i \leq k, \quad (1)$$

with initial boundary condition

$$z(t, x) = \varphi(t, x) \quad \text{on} \quad E_0 \cup \partial_0 E. \quad (2)$$

We will say that f and g satisfy the *condition* (V) if for each $(t, x) \in E$ and for $w, \tilde{w} \in C(B, \mathbb{R}^k)$ such that $w(\tau, y) = \tilde{w}(\tau, y)$ for $(\tau, y) \in D[t, x]$ we have $f(t, x, w) = f(t, x, \tilde{w})$ and $g(t, x, w) = g(t, x, \tilde{w})$. Note that the condition (V) means that the values of f and g at the point $(t, x, w) \in E \times C(B, \mathbb{R}^k)$ depend on (t, x) and on the restrictions of w to the set $D[t, x]$ only.

A function $v : E_0 \cup E \rightarrow \mathbb{R}^k$ is a classical solution of (1), (2) if

- (i) $v \in C(E_0 \cup E, \mathbb{R}^k)$ and the partial derivatives $\partial_t v_i$,

$$\partial_x v_i = (\partial_{x_1} v_i, \dots, \partial_{x_n} v_i), \quad 1 \leq i \leq k,$$

exist on E ,

- (ii) v satisfies equation (1) on E and condition (2) on $E_0 \cup \partial_0 E$.

The existence and uniqueness theorems for classical solutions of (1), (2) are based on two types of assumptions:

1. Regularity of given functions. The function f and g are assumed to be continuous and satisfy nonlinear estimates of the Perron type with respect to the functional variable.
2. Assumptions connected with the theory of bicharacteristics. It is assumed that

$$x_j f_{ij}(t, x, z) \geq 0, \quad 1 \leq j \leq n, \quad (t, x, z) \in E \times C(B, \mathbb{R}^k), \quad (3)$$

where $1 \leq i \leq k$. This assumption ensures that bicharacteristics of (1) satisfy the following monotonicity conditions. Suppose that $v \in C^1(E_0 \cup E, \mathbb{R}^k)$ and let the function $g_i[v](\cdot, t, x) = (g_{i1}[v](\cdot, t, x), \dots, g_{in}[v](\cdot, t, x))$, $(t, x) \in E$, denotes the solution of the Cauchy problem

$$\theta'(\tau) = -f_i(\tau, \theta(\tau), v), \quad \theta(t) = x.$$

The function $g_i[v](\cdot, t, x)$ is the i -th bicharacteristic of (1) corresponding to the solution v and starts at the point (t, x) . The condition (3) implies that for $0 \leq x_j \leq b_j$ the bicharacteristic $g_{ij}(\cdot, t, x)$ is non increasing and is nondecreasing for $-b_j \leq x_j < 0$.

The monotonicity property of bicharacteristics, which is obtained through the condition (3) and assumption on the regularity of given functions, ensures the existence of classical solutions for (1), (2). This results are based on the method of bicharacteristics and can be deduced from the monograph [16], Chapter V. The uniqueness criteria for (1), (2) can be received from comparison theorems for functional differential inequalities with initial boundary conditions ([1], [16]).

We are interested in numerical approximation of classical solutions to problem (1), (2).

Hyperbolic first order partial functional differential equations find applications in different branches of knowledge. The authors of [4] proposed quasilinear differential integral systems as simple mathematical models for the nonlinear phenomenon of harmonic generation of laser radiation through piezoelectric crystals for nondispersive materials and of Maxwell-Hopkinson type. Almost linear differential integral equations can be used to describe a model of proliferating cell population, see [6]. Quasilinear evolution equations with a bounded delay with applications to heat flow were considered in [5]. Hyperbolic conservation laws with finding memory can be viewed as quasilinear systems with integral terms of the Voltera type like we can observe in the paper [13].

The paper is organized as follows. In Section 2 we propose a new functional difference method corresponding to (1), (2). In Section 3 we prove that there is exactly one solution of the initial boundary value problem for difference equations generated by (1), (2). We give estimates of solutions to functional differential and functional difference problems. A convergence result and an error estimate of approximate solutions are presented in Section 4. Numerical examples are given in the last part of the paper.

2. Discretization of differential equations

We formulate a class of difference schemes for (1), (2). We will denote by $\mathbf{F}(X, Y)$ the class of all functions defined on X and taking values in Y , where X and Y are arbitrary sets. Let \mathbb{N} and \mathbb{Z} be the sets of natural numbers and

integers, respectively. For $x \in \mathbb{R}^n$, $p \in \mathbb{R}^k$ and for the matrix $U \in \mathbb{R}^{k \times n}$ we write

$$\begin{aligned} x &= (x_1, \dots, x_n), \quad \|x\| = \sum_{i=1}^n |x_i|, \\ p &= (p_1, \dots, p_k), \quad \|p\|_\infty = \max \{ |p_i| : 1 \leq i \leq k \}, \\ U &= [u_{ij}]_{i=1, \dots, k; j=1, \dots, n}, \quad \|U\| = \max \left\{ \sum_{j=1}^n |u_{ij}| : 1 \leq i \leq k \right\}. \end{aligned}$$

For a function $w \in C(B, \mathbb{R}^k)$ we define

$$\|w\|_B = \max \{ \|w(\tau, s)\|_\infty : (\tau, s) \in B \}.$$

We define a mesh on the set $E \cup E_0$ in the following way. Let (h_0, h') , $h' = (h_1, \dots, h_n)$, $h_j > 0$ for $0 \leq j \leq n$, stand for steps of the mesh. Let us denote by H the set of all $h = (h_0, h')$ such that there are $K_0 \in \mathbb{Z}$ and $K = (K_1, \dots, K_n) \in \mathbb{N}^n$ with the properties $K_0 h_0 = b_0$ and $(K_1 h_1, \dots, K_n h_n) = b$. For $h \in H$ and $(r, m) \in \mathbb{Z}^{1+n}$, where $m = (m_1, \dots, m_n)$, we define nodal points as follows

$$\begin{aligned} t^{(r)} &= r h_0, \\ x^{(m)} &= (x_1^{(m_1)}, \dots, x_n^{(m_n)}) = (m_1 h_1, \dots, m_n h_n). \end{aligned}$$

Let $N_0 \in \mathbb{N}$ be defined by the relations $N_0 h_0 \leq a < (N_0 + 1) h_0$. Write

$$R_h^{1+n} = \{ (t^{(r)}, x^{(m)}) : (r, m) \in \mathbb{Z}^{1+n} \}$$

and

$$\begin{aligned} E_h &= E \cap R_h^{1+n}, \quad E_{h,0} = E_0 \cap R_h^{1+n}, \\ \partial_0 E_h &= \partial_0 E \cap R_h^{1+n}, \quad B_h = B \cap R_h^{1+n}. \end{aligned}$$

Moreover we put

$$\begin{aligned} E_{h,r} &= (E_{h,0} \cup E_h) \cap \left([-b_0, t^{(r)}] \times \mathbb{R}^n \right), \quad -K_0 \leq r \leq N_0, \\ E'_h &= \left\{ (t^{(r)}, x^{(m)}) \in E_h \setminus \partial_0 E_h : 0 \leq r \leq N_0 - 1 \right\}, \\ I_h &= \left\{ t^{(r)} : -K_0 \leq r \leq N_0 \right\}, \quad I'_h = I_h \setminus \{ t^{(N_0)} \}. \end{aligned}$$

For a function $z : E_{h,0} \cup E_h \rightarrow \mathbb{R}^k$ we write $z^{(r,m)} = z(t^{(r)}, x^{(m)})$. Let $e_j = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{R}^n$, 1 standing on the j -th place, $1 \leq j \leq n$ and $0_{[n]} = (0, \dots, 0) \in \mathbb{R}^n$.

Since equation (1) contains the functional variable $z_{(t,x)}$ which is an element of the space $C(D[t, x], \mathbb{R}^k)$ then we use an interpolating operator

$T_h : \mathbf{F}(E_{0,h} \cup E_h, \mathbb{R}^k) \rightarrow C(E_0 \cup E, \mathbb{R}^k)$. For $z \in \mathbf{F}(E_{0,h} \cup E_h, \mathbb{R}^k)$ we write $(T_h z)_{[r,m]}$ instead of $(T_h z)_{(t^{(r)}, x^{(m)})}$. Appropriate assumptions on the operator T_h will be given in Section 3.

Suppose that the function $\varphi_h : E_{h,0} \cup \partial_0 E_h \rightarrow \mathbb{R}^k$, $\varphi_h = (\varphi_{h,1}, \dots, \varphi_{h,k})$ is given. Write

$$\delta_0 z = (\delta_0 z_1, \dots, \delta_0 z_k), \quad F[z]^{(r,m)} = \left(F_1[z]^{(r,m)}, \dots, F_k[z]^{(r,m)} \right)$$

and

$$F_i[z]^{(r,m)} = \sum_{j=1}^n f_{ij} \left(t^{(r)}, x^{(m)}, (T_h z)_{[r,m]} \right) \left[s_{ij} \delta_j z_i^{(r,m)} + (1 - s_{ij}) \delta_j z_i^{(r+1,m)} \right] \\ + g_i \left(t^{(r)}, x^{(m)}, (T_h z)_{[r,m]} \right), \quad i = 1, \dots, k,$$

where $0 \leq s_{ij} \leq 1$, $1 \leq i \leq k$, $1 \leq j \leq n$ are given constants. We consider the difference functional system

$$\delta_0 z^{(r,m)} = F[z]^{(r,m)} \quad (4)$$

with initial boundary condition

$$z^{(r,m)} = \varphi_h^{(r,m)} \quad \text{on} \quad E_{h,0} \cup \partial_0 E_h. \quad (5)$$

The difference operators δ_0 and $\delta = (\delta_1, \dots, \delta_n)$ are defined in the following way. Put

$$\delta_0 z^{(r,m)} = \frac{1}{h_0} \left(z^{(r+1,m)} - z^{(r,m)} \right). \quad (6)$$

If $f_{ij} \left(t^{(r)}, x^{(m)}, (T_h z)_{[r,m]} \right) \geq 0$ then

$$\delta_j z_i^{(r,m)} = \frac{1}{h_j} \left(z_i^{(r,m+e_j)} - z_i^{(r,m)} \right) \quad (7)$$

and

$$\delta_j z_i^{(r+1,m)} = \frac{1}{h_j} \left(z_i^{(r+1,m+e_j)} - z_i^{(r+1,m)} \right). \quad (8)$$

If $f_{ij} \left(t^{(r)}, x^{(m)}, (T_h z)_{[r,m]} \right) < 0$ then

$$\delta_j z_i^{(r,m)} = \frac{1}{h_j} \left(z_i^{(r,m)} - z_i^{(r,m-e_j)} \right) \quad (9)$$

and

$$\delta_j z_i^{(r+1,m)} = \frac{1}{h_j} \left(z_i^{(r+1,m)} - z_i^{(r+1,m-e_j)} \right). \quad (10)$$

We have $1 \leq i \leq k$ and $1 \leq j \leq n$ in (7)-(10)

Our difference functional problems have the following property: each equation in system (4) contains the parameters $s_i = (s_{i1}, \dots, s_{in})$, $1 \leq i \leq k$. If $s_i = (0, \dots, 0) \in \mathbb{R}^n$ for $1 \leq i \leq k$ then (4), (5) reduces to the explicit difference scheme. It is clear that there exists exactly one solution of problem (4), (5) in this case. The monograph [16] (Chapter V) contains sufficient conditions for the convergence of the explicit difference methods for first order partial differential equations.

Initial boundary value problem (4), (5) describes an implicit difference method in general case. We prove that under natural assumptions on f and g there exists exactly one solution $u_h : E_{h,0} \cup E_h \rightarrow \mathbb{R}^k$ to (4), (5). We give sufficient conditions for the convergence of implicit difference schemes. The proof of the stability of the methods is based on a comparison technique. It is important in our considerations that we assume nonlinear estimates of the Perron type for given functions with respect to the functional variable.

Note that if $k = 1$ and $s = (s_1, \dots, s_n) = (1, \dots, 1) \in \mathbb{R}^n$ then (4), (5) reduces to the implicit difference scheme considered in [18].

Difference schemes considered in the papers [24]- [26] depend on two parameters $s, \tilde{s} \in [0, 1]$. Right hand sides of difference equations corresponding to parabolic equations contain the expressions

$$s\delta z^{(r,m)} + (1-s)\delta z^{(r+1,m)} \quad \text{and} \quad \tilde{s}\delta^{(2)}z^{(r,m)} + (1-\tilde{s})\delta^{(2)}z^{(r+1,m)},$$

where $\delta = (\delta_1, \dots, \delta_n)$ and $\delta^{(2)} = [\delta_{ij}]_{i,j=1,\dots,n}$ are difference operators corresponding to the partial derivatives $\partial_x = (\partial_{x_1}, \dots, \partial_{x_n})$ and $\partial_{xx} = [\partial_{x_i x_j}]_{i,j=1,\dots,n}$.

3. Solutions of functional differential and difference equations

In this section we prove that there is exactly one solution of functional difference problem (4), (5). Moreover we give estimates of solutions to functional differential problem (1), (2) and of solutions to difference method (4), (5).

First we formulate a maximum principle for difference inequalities generated by (4), (5). Write

$$Y_h = \{m \in \mathbb{Z}^n : -b < x^m < b\}$$

and

$$J_{i,+}^{(r,m)}[z] = \{j : 1 \leq j \leq n \quad \text{and} \quad f_{ij}(t^{(r)}, x^{(m)}, (T_h z)_{[r,m]}) \geq 0\}, \quad (11)$$

$$J_{i,-}^{(r,m)}[z] = \{1, \dots, n\} \setminus J_{i,+}^{(r,m)}[z] \quad (12)$$

where $1 \leq i \leq k$.

Theorem 1 *Suppose that $0 \leq r \leq N_0 - 1$ is fixed and $z_h : E_{h,r} \rightarrow \mathbb{R}^k$ is known.*

(I) If $z_h : E_{h,r+1} \rightarrow \mathbb{R}^k$, $z_h = (z_{h,1}, \dots, z_{h,k})$, satisfies the difference inequalities

$$z_{h,i}^{(r+1,m)} \leq h_0 \sum_{j=1}^n f_{ij} \left(t^{(r)}, x^{(m)}, (T_h z_h)_{[r,m]} \right) (1 - s_{ij}) \delta_j z_{h,i}^{(r+1,m)}, \quad 1 \leq i \leq k,$$

for $m \in Y_h$ and $\mu^{(i)} \in \mathbb{Z}^n$, $\mu^{(i)} = (\mu_1^{(i)}, \dots, \mu_n^{(i)})$,

is such that $z_{h,i}^{(r+1,\mu^{(i)})} = M^{(i)}$ for $1 \leq i \leq k$, where

$$M^{(i)} = \max \left\{ z_{h,i}^{(r+1,m)} : -K < m < K \right\} \text{ and } M^{(i)} > 0, \quad (13)$$

then $(t^{(r+1)}, x^{(\mu^{(i)})}) \in \partial_0 E_h$.

(II) If $z_h : E_{h,r+1} \rightarrow \mathbb{R}^k$, $z_h = (z_{h,1}, \dots, z_{h,k})$, satisfies the difference inequalities

$$z_{h,i}^{(r+1,m)} \geq h_0 \sum_{j=1}^n f_{ij} \left(t^{(r)}, x^{(m)}, (T_h z_h)_{[r,m]} \right) (1 - s_{ij}) \delta_j z_{h,i}^{(r+1,m)}, \quad 1 \leq i \leq k,$$

for $m \in Y_h$ and $\tilde{\mu}^{(i)} \in \mathbb{Z}^n$, $\tilde{\mu}^{(i)} = (\tilde{\mu}_1^{(i)}, \dots, \tilde{\mu}_n^{(i)})$,

is such that $z_{h,i}^{(r+1,\tilde{\mu}^{(i)})} = \tilde{M}^{(i)}$ for $1 \leq i \leq k$, where

$$\tilde{M}^{(i)} = \min \left\{ z_{h,i}^{(r+1,m)} : -K < m < K \right\} \text{ and } \tilde{M}^{(i)} < 0,$$

then $(t^{(r+1)}, x^{(\tilde{\mu}^{(i)})}) \in \partial_0 E_h$. □

PROOF Consider the case (I). Suppose that i is fixed, $1 \leq i \leq k$, and that $(t^{(r+1)}, x^{(\mu^{(i)})}) \in E_h \setminus \partial_0 E_h$. Then using definitions (8), (10) of difference operators $\delta = (\delta_1, \dots, \delta_n)$ with z_h instead of z we have

$$\begin{aligned} & z_{h,i}^{(r+1,\mu^{(i)})} \\ \leq & h_0 \sum_{j \in J_{i,+}^{(r,m)}[z_h]} \frac{1}{h_j} f_{ij} \left(t^{(r)}, x^{(\mu^{(i)})}, (T_h z_h)_{[r,\mu^{(i)})} \right) (1 - s_{ij}) \left[z_{h,i}^{(r+1,\mu^{(i)}+e_j)} - z_{h,i}^{(r+1,\mu^{(i)})} \right] \\ + & h_0 \sum_{j \in J_{i,-}^{(r,m)}[z_h]} \frac{1}{h_j} f_{ij} \left(t^{(r)}, x^{(\mu^{(i)})}, (T_h z_h)_{[r,\mu^{(i)})} \right) (1 - s_{ij}) \left[z_{h,i}^{(r+1,\mu^{(i)})} - z_{h,i}^{(r+1,\mu^{(i)}-e_j)} \right]. \end{aligned}$$

This gives

$$z_{h,i}^{(r+1,\mu^{(i)})} \left[1 + h_0 \sum_{j=1}^n \frac{1}{h_j} (1 - s_{ij}) \left| f_{ij} \left(t^{(r)}, x^{(\mu^{(i)})}, (T_h z_h)_{[r,\mu^{(i)})} \right) \right| \right]$$

$$\begin{aligned} &\leq h_0 \sum_{j \in J_{i,+}^{(r,m)}[z_h]} \frac{1}{h_j} f_{ij} \left(t^{(r)}, x^{(\mu^{(i)})}, (T_h z_h)_{[r,\mu^{(i)}]} \right) (1 - s_{ij}) z_{h,i}^{(r+1,\mu^{(i)}+e_j)} \\ &- h_0 \sum_{j \in J_{i,-}^{(r,m)}[z_h]} \frac{1}{h_j} f_{ij} \left(t^{(r)}, x^{(\mu^{(i)})}, (T_h z_h)_{[r,\mu^{(i)}]} \right) (1 - s_{ij}) z_{h,i}^{(r+1,\mu^{(i)}-e_j)} \\ &\leq h_0 M^{(i)} \sum_{j=1}^n \frac{1}{h_j} (1 - s_{ij}) \left| f_{ij} \left(t^{(r)}, x^{(\mu^{(i)})}, (T_h z_h)_{[r,\mu^{(i)}]} \right) \right|. \end{aligned}$$

Then we get $z_{h,i}^{(r+1,\mu^{(i)})} \leq 0$, which contradicts (13). Then $(t^{(r+1)}, x^{(\mu^{(i)})}) \in \partial_0 E_h$ which is our claim. In a similar way we prove that $(t^{(r+1)}, x^{(\tilde{\mu}^{(i)})}) \in \partial_0 E_h$ for $1 \leq i \leq k$ in the case (II). This completes the proof. ■

Lemma 1 *Suppose that $f : E \times C(B, \mathbb{R}^k) \rightarrow \mathbb{R}^{k \times n}$, $g : E \times C(B, \mathbb{R}^k) \rightarrow \mathbb{R}^k$ and $h \in H$. Then difference functional problem (4), (5) with δ_0 and δ defined by (6)-(10) has exactly one solution $u_h : E_{h,0} \cup E_h \rightarrow \mathbb{R}^k$. □*

PROOF Suppose that $0 \leq r \leq N_0 - 1$ is fixed and $u_h : E_{h,r} \rightarrow \mathbb{R}^k$ is known. Then (4), (5) is the linear system from which we can calculate $u_h^{(r+1,m)}$ for $(t^{(r+1)}, x^{(m)}) \in E_h \setminus \partial_0 E_h$. The homogeneous problem corresponding to (4), (5) for $1 \leq i \leq k$ has the following form

$$z_i^{(r+1,m)} = h_0 \sum_{j=1}^n (1 - s_{ij}) f_{ij} \left(t^{(r)}, x^{(m)}, (T_h u_h)_{[r,m]} \right) \delta_j z_i^{(r+1,m)}, \quad (14)$$

$$z^{(r+1,m)} = 0 \quad \text{on} \quad E_{h,0} \cup \partial_0 E_h. \quad (15)$$

It follows from Theorem 1 that system (14), (15) has exactly one zero solution. Therefore the problem (4), (5) has exactly one solution. Then the numbers $u_h^{(r+1,m)}$ for $(t^{(r+1)}, x^{(m)}) \in E_h \setminus \partial_0 E_h$ exist and they are unique. Since u_h is given on $E_{h,0}$ then the proof is completed by induction. ■

We give estimates of solutions to (4), (5). For $z \in C(E_0 \cup E, \mathbb{R}^k)$ and $z_h \in \mathbf{F}(E_{h,0} \cup E_h, \mathbb{R}^k)$ we define the seminorms

$$\|z\|_t = \max \{ \|z(\tau, s)\|_\infty : (\tau, s) \in (E_0 \cup E) \cap ([-b_0, t] \times \mathbb{R}^n) \}, \quad 0 \leq t \leq a,$$

$$\|z_h\|_{h,r} = \max \{ \|z_h(\tau, s)\|_\infty : (\tau, s) \in E_{h,r} \}, \quad 0 \leq r \leq N_0.$$

We need the following assumptions.

Assumption $H[\varrho]$. The function $\varrho : [0, a] \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is continuous and

nondecreasing with respect to both variables. Moreover for each $\eta \in \mathbb{R}_+$ the maximal solution of the initial problem

$$\omega'(t) = \varrho(t, \omega(t)), \quad \omega(0) = \eta. \quad (16)$$

exists on $[0, a]$.

Assumption $H[f, g, \varphi]$. The functions $f : E \times C(B, \mathbb{R}^k) \rightarrow \mathbb{R}^{k \times n}$ and $g : E \times C(B, \mathbb{R}^k) \rightarrow \mathbb{R}^k$ are continuous and satisfy the condition (V) and

- 1) there is function $\varrho : [0, a] \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that Assumption $H[\varrho]$ is satisfied and

$$\|g(t, x, w)\|_\infty \leq \varrho(t, \|w\|_B) \text{ for } (t, x, w) \in E \times C(B, \mathbb{R}^k),$$

- 2) $\varphi \in C(E_0 \cup \partial_0 E, \mathbb{R}^k)$ and $\varphi_h \in \mathbf{F}(E_{h,0} \cup \partial_0 E_h, \mathbb{R}^k)$ and there is $\alpha_0 : \Delta \rightarrow \mathbb{R}_+$ such that

$$\left\| \varphi^{(r,m)} - \varphi_h^{(r,m)} \right\|_\infty \leq \alpha_0(h) \text{ on } E_{h,0} \cup \partial_0 E_h \text{ and } \lim_{h \rightarrow 0} \alpha_0(h) = 0.$$

Remark 1 Suppose that Assumption $H[f, g, \varphi]$ is satisfied. Then there is $\bar{\eta} \in \mathbb{R}_+$ such that

$$\|\varphi(t, x)\|_\infty \leq \bar{\eta} \text{ on } E_0 \text{ and } \|\varphi(t, x)\|_\infty \leq \omega(t, \bar{\eta}) \text{ on } \partial_0 E,$$

where $\omega(\cdot, \bar{\eta})$ is the maximal solution to (16) with $\eta = \bar{\eta}$. Moreover, there is $\tilde{\eta} \in \mathbb{R}_+$ such that

$$\left\| \varphi_h^{(r,m)} \right\|_\infty \leq \tilde{\eta} \text{ on } E_{h,0} \text{ and } \left\| \varphi_h^{(r,m)} \right\|_\infty \leq \omega(t^{(r)}, \tilde{\eta}) \text{ on } \partial_0 E_h$$

where $\omega(\cdot, \tilde{\eta})$ is the maximal solution to (16) with $\eta = \tilde{\eta}$. \square

Lemma 2 If Assumption $H[f, g, \varphi]$ is satisfied and $\bar{z} : E_0 \cup E \rightarrow \mathbb{R}^k$ is a solution to (1), (2) and \bar{z} is of class C^1 then

$$\|\bar{z}(t, x)\|_\infty \leq \omega(t, \bar{\eta}) \quad (17)$$

where $\omega(\cdot, \bar{\eta})$ is a solution to (16) with $\eta = \bar{\eta}$ and $\bar{\eta}$ is defined in Remark 1. \square

PROOF Write $\xi(t) = \|\bar{z}\|_t$, $t \in [0, a]$. Let us denote by $\omega(\cdot, \bar{\eta}, \varepsilon)$ the maximal solution of the initial problem

$$\omega'(t) = \varrho(t, \omega(t)) + \varepsilon, \quad \omega(0) = \bar{\eta} + \varepsilon$$

where $\varepsilon > 0$. There is $\tilde{\varepsilon} > 0$ such that for $0 < \varepsilon < \tilde{\varepsilon}$ the solution $\omega(\cdot, \bar{\eta}, \varepsilon)$ is defined on $[0, a]$ and

$$\lim_{\varepsilon \rightarrow 0} \omega(\cdot, \bar{\eta}, \varepsilon) = \omega(\cdot, \bar{\eta}) \text{ uniformly on } [0, a].$$

We prove that

$$\xi(t) < \omega(t, \bar{\eta}, \varepsilon) \tag{18}$$

for $t \in [0, a]$ and $0 < \varepsilon < \tilde{\varepsilon}$. Suppose by contradiction that this inequality fails to be true. Then there is $\tilde{t} \in (0, a)$ such that for $t \in [0, \tilde{t})$ we have $\xi(t) < \omega(t, \bar{\eta}, \varepsilon)$ and $\xi(\tilde{t}) = \omega(\tilde{t}, \bar{\eta}, \varepsilon)$. Moreover there exists $\tilde{x} = (\tilde{x}_1, \dots, \tilde{x}_n) \in [-b, b]$ and $i \in \{1, \dots, k\}$ such that $\xi(\tilde{t}) = |\bar{z}_i(\tilde{t}, \tilde{x})|$. Then two possibilities can happen, either (i) $\bar{z}_i(\tilde{t}, \tilde{x}) = \omega(\tilde{t}, \bar{\eta}, \varepsilon)$ or (ii) $\bar{z}_i(\tilde{t}, \tilde{x}) = -\omega(\tilde{t}, \bar{\eta}, \varepsilon)$. Let us consider the case (i). Then we have

$$D_-\xi(\tilde{t}) \geq \omega'(\tilde{t}, \bar{\eta}, \varepsilon) \tag{19}$$

where D_- is the left-hand lower Dini derivative. It is clear that $\partial_x \bar{z}_i(\tilde{t}, \tilde{x}) = 0_{[n]}$ and consequently from Assumption $H[g, \varphi]$ we have

$$D_-\xi(\tilde{t}) \leq \partial_t \bar{z}_i(\tilde{t}, \tilde{x}) \leq \varrho(\tilde{t}, \omega(\tilde{t}, \bar{\eta}, \varepsilon)) < \omega'(\tilde{t}, \bar{\eta}, \varepsilon)$$

which contradicts (19). The case (ii) can be treated in a similar way. Hence estimate (18) follows. Letting ε tend to zero in (18) we obtain the estimation (17). This completes the proof. ■

Assumption $H[T_h]$. The operator $T_h : \mathbf{F}(E_{0,h} \cup E_h, \mathbb{R}^k) \rightarrow C(E_0 \cup E, \mathbb{R}^k)$ satisfies the conditions:

1) for $z, \tilde{z} \in \mathbf{F}(E_{0,h} \cup E_h, \mathbb{R}^k)$ we have

$$\|T_h[z] - T_h[\tilde{z}]\|_{t^{(r)}} \leq \|z - \tilde{z}\|_{h,r}, \quad 0 \leq r \leq K, \tag{20}$$

2) if $z : E_0 \cup E \rightarrow \mathbb{R}$ is of class C^1 then there is $\gamma_\star : \Delta \rightarrow \mathbb{R}_+$ such that

$$\|T_h[z_h] - z\|_t \leq \gamma_\star(h) \text{ for } t \in [0, a] \text{ and } \lim_{h \rightarrow 0} \gamma_\star(h) = 0, \tag{21}$$

where z_h is the restriction of z to the set $E_{0,h} \cup E_h$.

Lemma 3 *Suppose that Assumptions $H[f, g, \varphi]$ and $H[T_h]$ are satisfied and for $(t, x, w) \in E_h \times C(B, \mathbb{R}^k)$ we have*

$$1 - h_0 \sum_{j=1}^n \frac{1}{h_j} s_{ij} |f_{ij}(t, x, w)| \geq 0, \quad 1 \leq i \leq k. \tag{22}$$

Then if $u_h : E_{0,h} \cup E_h \rightarrow \mathbb{R}^k$ is a solution of (4), (5) we have

$$\|u_h^{(r,m)}\|_\infty \leq \omega(t^{(r)}, \tilde{\eta}) \text{ on } E_h \tag{23}$$

where $\omega(\cdot, \tilde{\eta})$ is a maximal solution to (16) for $\eta = \tilde{\eta}$ and $\tilde{\eta}$ is defined in Remark 1. □

PROOF We conclude from (4) and from definitions of difference operators (6)-(10) that

$$\begin{aligned}
 & u_{h,i}^{(r+1,m)} \left[1 + h_0 \sum_{j=1}^n \frac{1}{h_j} (1 - s_{ij}) \left| f_{ij} \left(t^{(r)}, x^{(m)}, (T_h u_h)_{[r,m]} \right) \right| \right] \tag{24} \\
 & = u_{h,i}^{(r,m)} \left[1 - h_0 \sum_{j=1}^n \frac{1}{h_j} s_{ij} \left| f_{ij} \left(t^{(r)}, x^{(m)}, (T_h u_h)_{[r,m]} \right) \right| \right] \\
 & + h_0 \sum_{j \in J_{i,+}^{(r,m)}[u_h]} \frac{1}{h_j} f_{ij} \left(t^{(r)}, x^{(m)}, (T_h u_h)_{[r,m]} \right) \left(s_{ij} u_{h,i}^{(r,m+e_j)} + (1 - s_{ij}) u_{h,i}^{(r+1,m+e_j)} \right) \\
 & - h_0 \sum_{j \in J_{i,-}^{(r,m)}[u_h]} \frac{1}{h_j} f_{ij} \left(t^{(r)}, x^{(m)}, (T_h u_h)_{[r,m]} \right) \left(s_{ij} u_{h,i}^{(r,m-e_j)} + (1 - s_{ij}) u_{h,i}^{(r+1,m-e_j)} \right) \\
 & + h_0 g_i \left(t^{(r)}, x^{(m)}, (T_h u_h)_{[r,m]} \right), \quad \left(t^{(r)}, x^{(m)} \right) \in E'_h.
 \end{aligned}$$

Let us define $\lambda_h : I_h \rightarrow \mathbb{R}_+$ by $\lambda_h^{(r)} = \|u_h\|_{h,r}$, $0 \leq r \leq N_0$. It follows from condition 1) of Assumption $H[f, g, \varphi]$ and from (24) that

$$\lambda_h^{(r+1)} \leq \lambda_h^{(r)} + h_0 \varrho(t^{(r)}, \lambda_h^{(r)}), \quad 0 \leq r \leq N_0 - 1. \tag{25}$$

Based on Remark 1 we have $\lambda_h^{(0)} \leq \tilde{\eta}$. The maximal solution $\omega(\cdot, \tilde{\eta})$ of (16) is a convex function therefore satisfies the recurrent difference inequality

$$\omega \left(t^{(r+1)}, \tilde{\eta} \right) \geq \omega \left(t^{(r)}, \tilde{\eta} \right) + h_0 \varrho \left(t^{(r)}, \omega(t^{(r)}, \tilde{\eta}) \right), \quad 0 \leq r \leq N_0 - 1.$$

It follows from above and from (25) that $\lambda_h^{(r)} \leq \omega(t^{(r)}, \tilde{\eta})$ for $0 \leq r \leq N_0$. This proves (23). This completes the proof. ■

Remark 2 The assumption (22) is called the Courant-Friedrichs-Lévy condition for problem (4)-(5) (see [14] Chapter III and [16] Chapter V). □

4. Convergence of difference methods Let $\eta_* = \max \{ \bar{\eta}, \tilde{\eta} \}$ where $\bar{\eta}$ and $\tilde{\eta}$ are defined in Remark 1. Set

$$\Omega_C = \left\{ (t, x, w) \in E \times C(B, \mathbb{R}^k) : \|w\|_B \leq C \right\}$$

where $C = \omega(a, \eta_*)$ and $\omega(\cdot, \eta_*)$ is a solution of (16) with $\eta = \eta_*$.

To prove the convergence of functional difference problem (4), (5) we need the following additional assumptions.

Assumption $H[f, g, \sigma]$. Suppose that

1) there is $\sigma : [0, a] \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that

- (i) σ is continuous and nondecreasing with respect to both variables,
- (ii) $\sigma(t, 0) = 0$ for $t \in [0, a]$ and for each $c \geq 1$ the maximal solution of the Cauchy problem

$$w'(t) = cw(t), \quad w(0) = 0, \tag{26}$$

is $\tilde{w}(t) = 0$ for $t \in (0, a)$,

2) for each $(t, x, w) \in E \times C(B, \mathbb{R}^k)$ we have

$$x_j f_{ij}(t, x, w) \geq 0, \quad 1 \leq j \leq n, \quad 1 \leq i \leq k,$$

3) the estimates

$$\|f(t, x, w) - f(t, x, \bar{w})\| \leq \sigma(t, \|w - \bar{w}\|_B), \tag{27}$$

$$\|g(t, x, w) - g(t, x, \bar{w})\|_\infty \leq \sigma(t, \|w - \bar{w}\|_B) \tag{28}$$

are satisfied on Ω_C .

Remark 3 It is important that we have assumed nonlinear estimates of Peron type (27) and (28) on Ω_C . There are differential equations with deviated variables and differential integral equations such that condition 2) of Assumption $H[f, g, \sigma]$ is satisfied and global estimates for f and g are not satisfied. We give comments on such equations. □

Suppose that the functions $\tilde{f} : E \times \mathbb{R}^k \rightarrow \mathbb{R}^{k \times n}$, $\tilde{f} = [\tilde{f}_{ij}]_{i=1, \dots, k, j=1, \dots, n}$, $\tilde{g} = (\tilde{g}_1, \dots, \tilde{g}_k) : E \times \mathbb{R}^k \rightarrow \mathbb{R}^k$ of the variables (t, x, p) are continuous and

- (i) there exist the derivatives $\partial_{p_i} \tilde{f} = (\partial_{p_i} \tilde{f}_1, \dots, \partial_{p_i} \tilde{f}_n)$, $\partial_{p_i} \tilde{g}$, $1 \leq i \leq k$, and $\partial_p \tilde{f} \in C(E \times \mathbb{R}^k, \mathbb{R}^{k \times n})$, $\partial_p \tilde{g} \in C(E \times \mathbb{R}^k, \mathbb{R}^k)$,
- (ii) the function $\partial_p \tilde{f}$ and $\partial_p \tilde{g}$ are unbounded on $E \times \mathbb{R}^k$ and there are $\tilde{\alpha}, \tilde{\beta} \in \mathbb{R}_+$ such that

$$\|\tilde{g}(t, x, p)\|_\infty \leq \tilde{\alpha} \|p\|_\infty + \tilde{\beta} \text{ on } E \times \mathbb{R}^k.$$

Assume that $\psi \in C(E, \mathbb{R}^{1+n})$, $\psi = (\psi_0, \psi_1, \dots, \psi_n)$, is a given function and $\psi(t, x) \in E$ for $(t, x) \in E$ and $\psi_0(t, x) \leq t$ for $(t, x) \in E$. Then $(\psi(t, x) - (t, x)) \in B$ for $(t, x) \in E$. Let $f : E \times C(B, \mathbb{R}^k) \rightarrow \mathbb{R}^{k \times n}$ and $g : E \times C(B, \mathbb{R}^k) \rightarrow \mathbb{R}^k$ be defined by

$$f(t, x, w) = \tilde{f}(t, x, w(\psi(t, x) - (t, x))), \quad g(t, x, w) = \tilde{g}(t, x, w(\psi(t, x) - (t, x))). \tag{29}$$

Then (1) reduces to the system of differential equations with deviated variables

$$\partial_t z_i(t, x) = \sum_{j=1}^n \tilde{f}_{ij}(t, x, z(\psi(t, x))) \partial_{x_j} z_i(t, x) + \tilde{g}(t, x, z(\psi(t, x))), \quad 1 \leq i \leq k.$$

It follows that there is $L \in \mathbb{R}_+$ such that the functions given by (29) satisfy Assumption $H[f, g, \sigma]$ with $\sigma(t, p) = Lp$, $(t, p) \in [0, a] \times \mathbb{R}_+$, and the global Lipschitz condition with respect to the functional variable is not satisfied.

For the above \tilde{f} and \tilde{g} we put

$$\begin{aligned} f(t, x, w) &= \tilde{f} \left(t, x, \int_{D[t, x]} w(\tau, s) d\tau ds \right), \\ g(t, x, w) &= \tilde{g} \left(t, x, \int_{D[t, x]} w(\tau, s) d\tau ds \right). \end{aligned} \quad (30)$$

Then (1) reduces to the system of differential integral equations

$$\begin{aligned} \partial_t z_i(t, x) &= \sum_{j=1}^n \tilde{f}_{ij} \left(t, x, \int_{D[t, x]} z(\tau, s) d\tau ds \right) \partial_{x_j} z_i(t, x) \\ &\quad + \tilde{g} \left(t, x, \int_{D[t, x]} z(\tau, s) d\tau ds \right), \quad 1 \leq i \leq k. \end{aligned}$$

It is clear that there is $L \in \mathbb{R}_+$ such that the functions given by (30) satisfy Assumption $H[f, g, \sigma]$ with $\sigma(t, p) = Lp$, $(t, p) \in [0, a] \times \mathbb{R}_+$, and the global Lipschitz condition with respect to the functional variable is not satisfied.

Now we conduct an analysis of the convergence of the difference method (4), (5).

Theorem 2 *Suppose that Assumptions $H[f, g, \sigma]$, $H[f, g, \varphi]$ and $H[T_h]$ are satisfied and*

1) $v : E_0 \cup E \rightarrow \mathbb{R}^k$ is a solution to (1), (2) and v is of class C^1 on $E_0 \cup E$ and v_h is the restriction of v to $E_{h,0} \cup E_h$,

2) for $(t, x, w) \in E \times C(B, \mathbb{R}^k)$ we have

$$1 - h_0 \sum_{j=1}^n \frac{1}{h_j} s_{ij} |f_{ij}(t, x, w)| \geq 0, \quad 1 \leq i \leq k, \quad (31)$$

3) there is $c_0 \in \mathbb{R}_+$ such that the following estimate

$$\left\| \delta_j v_h^{(r, m)} \right\|_{\infty} \leq c_0 \quad (32)$$

is satisfied for $1 \leq j \leq n$. □

Then there is exactly one solution $u_h : E_{h,0} \cup E_h \rightarrow \mathbb{R}^k$ to (1), (2) and there is $\alpha : H \rightarrow \mathbb{R}_+$ such that

$$\left\| v_h^{(r,m)} - u_h^{(r,m)} \right\|_\infty \leq \alpha(h) \quad \text{and} \quad \lim_{h \rightarrow 0} \alpha(h) = 0. \quad (33)$$

PROOF It follows from Lemma 1 that there is exactly one solution to (4), (5). We prove (33). Let $\Gamma_h : E'_h \rightarrow \mathbb{R}^k$ be defined by the relation

$$\delta_0 v_h^{(r,m)} = F[v_h]^{(r,m)} + \Gamma_h^{(r,m)}. \quad (34)$$

There is $\gamma : H \rightarrow \mathbb{R}_+$ such that

$$\left\| \Gamma_h^{(r,m)} \right\|_\infty \leq \gamma(h) \quad \text{on} \quad E'_h \quad \text{and} \quad \lim_{h \rightarrow 0} \gamma(h) = 0. \quad (35)$$

Put $P[z]^{(r,m)} = (t^{(r)}, x^{(m)}, (T_h z)_{[r,m]})$. Write $\phi_h = v_h - u_h$ then from (4) and (34) we have

$$\begin{aligned} \phi_{h,i}^{(r+1,m)} &= \phi_{h,i}^{(r)} + h_0 \sum_{j=1}^n f_{ij} \left(P[u_h]^{(r,m)} \right) \left[s_{ij} \delta_j \phi_{h,i}^{(r,m)} \right. \\ &\quad \left. + (1 - s_{ij}) \delta_j \phi_{h,i}^{(r+1,m)} \right] + \Lambda_{h,i}^{(r,m)} \end{aligned} \quad (36)$$

where $1 \leq i \leq k$ and

$$\begin{aligned} \Lambda_{h,i}^{(r,m)} &= \\ &= h_0 \sum_{j=1}^n \left(f_{ij} \left(P[v_h]^{(r,m)} \right) - f_{ij} \left(P[u_h]^{(r,m)} \right) \right) \left[s_{ij} \delta_j v_{h,i}^{(r,m)} + (1 - s_{ij}) \delta_j v_{h,i}^{(r+1,m)} \right] \\ &\quad + h_0 \left(g_i \left(P[v_h]^{(r,m)} \right) - g_i \left(P[u_h]^{(r,m)} \right) \right) + h_0 \Gamma_{h,i}^{(r,m)}. \end{aligned}$$

From above and from (7)-(10) we get

$$\begin{aligned} \phi_{h,i}^{(r+1,m)} &\left[1 + h_0 \sum_{j=1}^n \frac{1}{h_j} (1 - s_{ij}) \left| f_{ij} \left(P[u_h]^{(r,m)} \right) \right| \right] \\ &= \phi_{h,i}^{(r,m)} \left[1 - h_0 \sum_{i=1}^n \frac{1}{h_j} s_{ij} \left| f_{ij} \left(P[u_h]^{(r,m)} \right) \right| \right] \\ &\quad + h_0 \sum_{j \in J_{i,+}^{(r,m)}[u_h]} \frac{1}{h_j} f_{ij} \left(P[u_h]^{(r,m)} \right) \left[s_{ij} \phi_{h,i}^{(r,m+e_j)} + (1 - s_{ij}) \phi_{h,i}^{(r+1,m+e_j)} \right] \\ &\quad - h_0 \sum_{j \in J_{i,-}^{(r,m)}[u_h]} \frac{1}{h_j} f_{ij} \left(P[u_h]^{(r,m)} \right) \left[s_{ij} \phi_{h,i}^{(r,m-e_j)} + (1 - s_{ij}) \phi_{h,i}^{(r+1,m-e_j)} \right] + \Lambda_{h,i}^{(r,m)} \end{aligned} \quad (37)$$

where $J_{i,+}^{(r,m)}[u_h]$ and $J_{i,-}^{(r,m)}[u_h]$ are defined by (11), (12). Set $\lambda_h^{(r)} = \|\phi_h\|_{h,r}$, $0 \leq r \leq N_0$. Then it follows from Assumptions $H[T_h]$, $H[f, g, \sigma]$ and the assumptions of Theorem 4.1 that λ_h satisfies the recurrent inequality

$$\lambda_h^{(r+1)} \leq \lambda_h^{(r)} + h_0(1 + nc_0)\sigma(t^{(r)}, \lambda_h^{(r)}) + h_0\gamma_h, \quad 1 \leq r \leq N_0 - 1. \quad (38)$$

From condition 2) of Assumption $H[f, g, \varphi]$ we have $\lambda_h^{(0)} \leq \alpha_0(h)$. Let us denote by $\omega(\cdot, h)$ the maximal solution of the following initial problem

$$\omega'(t) = (1 + nc_0)\sigma(t, \omega(t)) + \gamma(h), \quad \omega(0) = \alpha_0(h). \quad (39)$$

Then $\omega(\cdot, h)$ is defined on $[0, a]$ and

$$\lim_{h \rightarrow 0} \omega(t, h) = 0 \quad \text{uniformly on } [0, a].$$

It follows from condition 1) of Assumption $H[f, g, \sigma]$ that $\omega(\cdot, h)$ is convex and satisfies the recurrent difference inequality

$$\omega(t^{(r+1)}, h) \geq \omega(t^{(r)}, h) + h_0\sigma(t^{(r)}, \omega(t^{(r)}, h)) + h_0\gamma(h),$$

where $1 \leq r \leq N_0 - 1$. From above and from (38) we get

$$\lambda_h^{(r)} \leq \omega(t^{(r)}, h), \quad 1 \leq r \leq N_0.$$

Then the condition (33) is satisfied with $\alpha(h) = \omega(a, h)$. This completes the proof. \blacksquare

Remark 4 The classical solutions of the functional differential problem (1), (2) are approximate solutions to the problem (4), (5). Then the assumption (32) of Theorem 2 is satisfied. \square

Now we give error estimate for difference method (4), (5). First we introduce an example of the operator T_h satisfying Assumption $H[T_h]$. Put

$$S_* = \{(j, s) : j \in \{0, 1\}, s = (s_1, \dots, s_n), s_i \in \{0, 1\} \text{ for } 1 \leq i \leq n\}.$$

Let $w \in F(E_{h,0} \cup E_h, \mathbb{R}^k)$ and $(t, x) \in E_0 \cup E$. There exists $(t^{(r)}, x^{(m)}) \in E_{h,0} \cup E_h$ such that

$$t^{(r)} \leq t \leq t^{(r+1)}, \quad x^{(m)} \leq x \leq x^{(m+1)}, \quad (t^{(r+1)}, x^{(m+1)}) \in E_{h,0} \cup E_h.$$

We define

$$\begin{aligned} (T_h w)(t, x) &= \sum_{(j,s) \in S_*} w^{(r+j,m+s)} \left(\frac{Y - Y^{(r,m)}}{h} \right)^{(j,s)} \\ &\quad \times \left(1 - \frac{Y - Y^{(r,m)}}{h} \right)^{1-(j,s)} \end{aligned}$$

where

$$\left(\frac{Y - Y^{(r,m)}}{h}\right)^{(j,s)} = \left(\frac{t - t^{(r)}}{h_0}\right)^j \prod_{k=1}^n \left(\frac{x_k - x_k^{(m_k)}}{h_k}\right)^{s_k}$$

and

$$\left(1 - \frac{Y - Y^{(r,m)}}{h}\right)^{1-(j,s)} = \left(1 - \frac{t - t^{(r)}}{h_0}\right)^{1-j} \prod_{k=1}^n \left(1 - \frac{x_k - x_k^{(m_k)}}{h_k}\right)^{1-s_k}$$

and we take $0^0 = 1$ in the above formulas. It is easy to see that $T_h w \in C(E_0 \cup E, \mathbb{R}^k)$. The above interpolating operator has been defined in [16], Chapter 5.

Theorem 3 *Suppose that*

- 1) *all assumptions of Theorem 2 are satisfied with $\sigma(t, p) = Lp$ and the solution $v : E_0 \cup E \rightarrow \mathbb{R}^k$ of differential problem (1), (2) is of class C^2 ,*
- 2) *the constant $\tilde{C} > 0$ is such that*

$$\|\partial_t v(t, x)\|_\infty, \quad \|\partial_{x_j} v(t, x)\|_\infty \leq \tilde{C} \quad \text{on } E_0 \cup E,$$

$$\|\partial_{tt} v(t, x)\|_\infty \leq \tilde{C}, \quad \|\partial_{tx_j} v(t, x)\|_\infty \leq \tilde{C},$$

$$\|\partial_{x_j x_k} v(t, x)\|_\infty \leq \tilde{C} \quad \text{on } E_0 \cup E,$$

where $1 \leq j, k \leq n$, and there exists $\tilde{d} \in \mathbb{R}_+$ such that

$$\|f(t, x, w)\| \leq \tilde{d} \quad \text{on } E \times C(B, \mathbb{R}^k).$$

Then

$$\|u_h^{(r,m)} - v_h^{(r,m)}\|_\infty \leq \bar{\alpha}(h) \quad \text{on } E_h \tag{40}$$

where v_h is a restriction of v to $E_{h,0} \cup E_h$ and

$$\bar{\alpha}(h) = \alpha_0(h)e^{L\tilde{C}a} + \gamma(h) \frac{e^{L\tilde{C}a} - 1}{L\tilde{C}},$$

$$\gamma(h) = \tilde{C} \left[\frac{1}{2}h_0 + (1 + \|h'\|) \tilde{d} \right] + L\tilde{C}(1 + \tilde{C})\|h\|^2$$

where $h = (h_0, h') = (h_0, h_1, \dots, h_n)$. □

PROOF From assumptions of the Theorem we conclude that the difference operators δ_0 and δ satisfy the conditions

$$\|\delta_0 v_h^{(r,m)} - \partial_t v^{(r,m)}\|_\infty \leq \frac{1}{2}\tilde{C}h_0, \tag{41}$$

$$\left\| \delta_j v_h^{(r,m)} - \partial_{x_j} v^{(r,m)} \right\|_\infty \leq \frac{1}{2} \tilde{C} \|h'\|, \quad 1 \leq j \leq n. \tag{42}$$

We have

$$\begin{aligned} \Gamma_{h,i}^{(r,m)} &= \delta_0 v_{h,i}^{(r,m)} - \partial_t v_i^{(r,m)} \\ &+ \sum_{j=1}^n f_{ij} \left(t^{(r)}, x^{(m)}, (T_h v_h)_{[r,m]} \right) \left[s_{ij} \delta_j v_{h,i}^{(r,m)} + (1 - s_{ij}) \delta_j v_{h,i}^{(r+1,m)} \right] \\ &- \sum_{j=1}^n f_{ij} \left(t^{(r)}, x^{(m)}, v_{(t^{(r)}, x^{(m)})} \right) \partial_{x_j} v_i^{(r,m)} \\ &+ g_i \left(t^{(r)}, x^{(m)}, (T_h v_h)_{[r,m]} \right) - g_i \left(t^{(r)}, x^{(m)}, v_{(t^{(r)}, x^{(m)})} \right), \quad 1 \leq i \leq k. \end{aligned}$$

It follows from Theorem 5.27 in [16] that there is $\tilde{C} \in \mathbb{R}_+$ such that

$$\|T_h v_h - v\|_B \leq \tilde{C} \|h\|^2. \tag{43}$$

From Assumption $H[f, g, \sigma]$ and from estimates (41)-(43) we get

$$\|\Gamma_h^{(r,m)}\|_\infty \leq \gamma(h).$$

Then the inequality (40) is obtained by solving problem (39) with $\sigma(t, p) = Lp$. This completes the proof. ■

5. Numerical examples

Example 1 For $n = 1$ and $k = 2$ we define

$$E = [0, 0.25] \times [-1, 1], \quad E_0 = \{0\} \times [-1, 1].$$

Consider quasilinear system of differential integral equations with deviated variables

$$\begin{aligned} \partial_t z_1(t, x) &= \left[1 + \sin \left(t e^t \int_0^x z_1(t, \tau) d\tau + t \int_0^x z_2(t, \tau) d\tau \right. \right. \\ &\quad \left. \left. - e^{tx} + e^{-tx} \right) \right] \partial_x z_1(t, x) \\ &+ z_1(t, 0.5x) \cos \left((x - 1) \int_0^t z_1(\tau, x) d\tau + 1 - z_1(t, x) \right) \\ &+ z_1(t, x) \left(x - t - 1 - z_2(t, x) e^{0.5tx} \right), \end{aligned} \tag{44}$$

$$\begin{aligned} \partial_t z_2(t, x) &= \left[1 + \cos \left(z_1(t, 0.5x) z_2(0.5t, x) - e^{-t} \right) \right] \partial_x z_2(t, x) \\ &+ x \int_0^t z_2(\tau, x) d\tau \sin \left(z_2(0.5t, x) - e^{-0.5tx} + \frac{\pi}{2} \right) \\ &+ z_2(t, x) (2t - x + 1) - 1 \end{aligned} \tag{45}$$

with initial boundary conditions

$$\begin{aligned} (z_1(0, x), z_2(0, x)) &= (1, 1), & x \in [-1, 1], \\ (z_1(t, 1), z_2(t, 1)) &= (1, e^t), & t \in [0, 0.25]. \end{aligned} \quad (46)$$

The exact solution of this problem is known. It is $z(t, x) = (z_1(t, x), z_2(t, x)) = (e^{t(x-1)}, e^{-tx})$. \square

To approximate solutions of the above differential problem we consider the following discretization of equations (44), (45)

$$\begin{aligned} z_1^{(r+1,m)} &= z_1^{(r,m)} + h_0 \left[1 + \sin \left(t^{(r)} e^{t^{(r)}} \int_0^{x^{(m)}} z_1(t^{(r)}, \tau) d\tau \right. \right. \\ &\quad \left. \left. + t^{(r)} \int_0^{x^{(m)}} z_2(t^{(r)}, \tau) d\tau - e^{t^{(r)}x^{(m)}} + e^{-t^{(r)}x^{(m)}} \right) \right] \\ &\quad \times \left(s_{11} \delta z_1^{(r,m)} + (1 - s_{11}) \delta z_1^{(r+1,m)} \right) \\ &\quad + z_1(t^{(r)}, 0.5x^{(m)}) \\ &\quad \times \cos \left((x^{(m)} - 1) \int_0^{t^{(r)}} z_1(\tau, x^{(m)}) d\tau + 1 - z_1^{(r,m)} \right) \\ &\quad + z_1^{(r,m)} \left(x^{(m)} - t^{(r)} - 1 - z_2(t^{(r)}, x^{(m)}) e^{0.5t^{(r)}x^{(m)}} \right), \end{aligned} \quad (47)$$

$$\begin{aligned} z_2^{(r+1,m)} &= z_2^{(r,m)} + h_0 \left[1 + \cos \left(z_1(t^{(r)}, 0.5x^{(m)}) z_2(0.5t^{(r)}, x^{(m)}) \right. \right. \\ &\quad \left. \left. - e^{-t^{(r)}} \right) \right] \left(s_{21} \delta z_2^{(r,m)} + (1 - s_{21}) \delta z_2^{(r+1,m)} \right) \\ &\quad + x^{(r)} \int_0^{t^{(r)}} z_2(\tau, x^{(m)}) d\tau \\ &\quad \times \sin \left(z_2(0.5t^{(r)}, x^{(m)}) - e^{-0.5t^{(r)}x^{(m)}} + \frac{\pi}{2} \right) \\ &\quad + z_2^{(r,m)} (2t^{(r)} - x^{(m)} + 1) - 1, \end{aligned} \quad (48)$$

with a discrete initial boundary condition corresponding to (46). For a simplicity let us put $s = s_{11} = s_{21}$. If we take $s = 1$ in (47) and (48), we get explicit difference method. If $s = 0$ we will say that we have strong implicit difference method.

Let us denote by $z_h = (z_{h,1}, z_{h,2})$ the solution to (47), (48) with a discrete initial boundary condition. The following tables show maximal values of errors for several step sizes with respect to the value of parameter s .

Note that for steps which satisfy the CFL condition (Table 1) explicit method gives the best results and the strong implicit method gives the worse results.

Table 1: Maximal values of errors

(h_0, h_1)	z_h	$s = 1$	$s = 0.75$	$s = 0.5$	$s = 0$
$(2^{-8}, 2^{-6})$	$z_{h.1}$	$4.30455 \cdot 10^{-3}$	$4.47828 \cdot 10^{-3}$	$4.65205 \cdot 10^{-3}$	$4.99970 \cdot 10^{-3}$
	$z_{h.2}$	$3.00355 \cdot 10^{-4}$	$1.03621 \cdot 10^{-3}$	$1.77098 \cdot 10^{-3}$	$3.23726 \cdot 10^{-3}$
$(2^{-10}, 2^{-9})$	$z_{h.1}$	$5.18508 \cdot 10^{-3}$	$5.22908 \cdot 10^{-3}$	$5.27309 \cdot 10^{-3}$	$5.36111 \cdot 10^{-3}$
	$z_{h.2}$	$8.64527 \cdot 10^{-5}$	$2.78026 \cdot 10^{-4}$	$4.69525 \cdot 10^{-4}$	$8.52301 \cdot 10^{-4}$
$(2^{-12}, 2^{-9})$	$z_{h.1}$	$5.44808 \cdot 10^{-3}$	$5.45912 \cdot 10^{-3}$	$5.47017 \cdot 10^{-3}$	$5.49226 \cdot 10^{-3}$
	$z_{h.2}$	$1.12365 \cdot 10^{-5}$	$5.95956 \cdot 10^{-5}$	$1.07950 \cdot 10^{-4}$	$2.04645 \cdot 10^{-4}$

Table 2: Maximal values of errors, violated CFL condition

(h_0, h_1)	z_h	$s = 1$	$s = 0.75$	$s = 0.5$	$s = 0$
$(2^{-5}, 2^{-8})$	$z_{h.1}$	$5.21595 \cdot 10^0$	$3.63995 \cdot 10^{-3}$	$3.11675 \cdot 10^{-3}$	$5.95879 \cdot 10^{-3}$
	$z_{h.2}$	$2.48934 \cdot 10^3$	$7.50288 \cdot 10^{-3}$	$1.05245 \cdot 10^{-2}$	$1.82619 \cdot 10^{-3}$
$(2^{-6}, 2^{-8})$	$z_{h.1}$	$4.34715 \cdot 10^{26}$	$5.65002 \cdot 10^{-3}$	$1.83288 \cdot 10^{-3}$	$3.56439 \cdot 10^{-3}$
	$z_{h.2}$	$2.39661 \cdot 10^{10}$	$3.46226 \cdot 10^{-1}$	$6.46328 \cdot 10^{-3}$	$1.14059 \cdot 10^{-2}$
$(2^{-8}, 2^{-10})$	$z_{h.1}$	∞	$6.98941 \cdot 10^{175}$	$4.47489 \cdot 10^{-3}$	$4.82136 \cdot 10^{-3}$
	$z_{h.2}$	∞	$1.99341 \cdot 10^{12}$	$1.86107 \cdot 10^{-3}$	$3.32952 \cdot 10^{-3}$
$(2^{-9}, 2^{-11})$	$z_{h.1}$	∞	∞	$4.99253 \cdot 10^{-3}$	$5.16759 \cdot 10^{-3}$
	$z_{h.2}$	∞	∞	$9.51809 \cdot 10^{-4}$	$1.70690 \cdot 10^{-3}$

In the case when the CFL condition is violated (table 2) the explicit difference scheme is divergent, implicit difference method with $s = 0.75$ is not stable. We get the best results for implicit difference scheme with $s = 0.5$.

Example 2 For $n = 2$ and $k = 1$ we define

$$\tilde{E} = [0, 0.25] \times [-1, 1] \times [-1, 1], \quad \tilde{E}_0 = \{0\} \times [-1, 1] \times [-1, 1].$$

Consider differential equation with deviated variables

$$\begin{aligned} \partial_t z(t, x, y) = & [1 + \cos(z(t, 0.5(x+y), 0.5(x-y)))] \partial_x z(t, x, y) \\ & - [1 + \sin(z(0.5t, 0.5x, 0.5y))] \partial_y z(t, x, y) + (z(0.5t, 0.5x, 0.5y))^8 \\ & + z(t, x, y) \left[xy - 1 - ty \left(1 + \cos \left(e^{0.25t(x^2-y^2)} \right) \right) + tx \left(1 + \sin \left(e^{0.125txy} \right) \right) \right] \end{aligned}$$

with initial boundary condition

$$z(0, x, y) = 1, \quad (x, y) \in [-1, 1] \times [-1, 1],$$

$$z(t, 1, y) = \cos(ty), \quad t \in [0, 0.25], \quad y \in [-1, 1],$$

$$z(t, x, -1) = \cos(tx), \quad t \in [0, 0.25], \quad x \in [-1, 1].$$

The exact solution of this problem is known. It is $z(t, x, y) = \cos(t(1-x+y))$. \square

In this case we have $i \in \{1\}$ and $j \in \{1, 2\}$ therefore in a difference method for above differential equation we approximate partial derivatives in the following way

$$\begin{aligned} \partial_x z(t, x, y) &\approx \frac{1}{h_1} \left[s_{11} \left(z^{(r, m_1+1, m_2)} - z^{(r, m_1, m_2)} \right) \right. \\ &\quad \left. + (1 - s_{11}) \left(z^{(r+1, m_1+1, m_2)} - z^{(r+1, m_1, m_2)} \right) \right], \\ \partial_y z(t, x, y) &\approx \frac{1}{h_2} \left[s_{12} \left(z^{(r, m_1, m_2)} - z^{(r, m_1, m_2-1)} \right) \right. \\ &\quad \left. + (1 - s_{12}) \left(z^{(r+1, m_1, m_2)} - z^{(r+1, m_1, m_2-1)} \right) \right]. \end{aligned}$$

In the given results for a simplicity we also adopted $s_{11} = s_{12} = s$. The following tables show maximal values of errors for several step sizes with respect to the value of parameter s . From Tables 3 and 4 we get the same

Table 3: Maximal values of errors

(h_0, h_1, h_2)	$s = 1$	$s = 0.75$	$s = 0.5$	$s = 0$
$(2^{-6}, 2^{-4}, 2^{-4})$	$4.38177 \cdot 10^{-4}$	$2.05186 \cdot 10^{-3}$	$3.79256 \cdot 10^{-3}$	$7.17073 \cdot 10^{-3}$
$(2^{-9}, 2^{-6}, 2^{-6})$	$5.22318 \cdot 10^{-5}$	$3.01896 \cdot 10^{-4}$	$5.86684 \cdot 10^{-4}$	$1.15029 \cdot 10^{-3}$
$(2^{-10}, 2^{-7}, 2^{-7})$	$2.68465 \cdot 10^{-5}$	$1.77695 \cdot 10^{-4}$	$3.07460 \cdot 10^{-4}$	$6.02534 \cdot 10^{-4}$

Table 4: Maximal values of errors, violated CFL condition

(h_0, h_1, h_2)	$s = 1$	$s = 0.75$	$s = 0.5$	$s = 0$
$(2^{-5}, 2^{-7}, 2^{-7})$	$2.68595 \cdot 10^2$	$7.85583 \cdot 10^{-3}$	$8.19253 \cdot 10^{-3}$	$1.40493 \cdot 10^{-2}$
$(2^{-6}, 2^{-8}, 2^{-8})$	$5.01513 \cdot 10^{12}$	$1.12682 \cdot 10^0$	$4.85345 \cdot 10^{-3}$	$8.56825 \cdot 10^{-3}$
$(2^{-7}, 2^{-9}, 2^{-9})$	$2.12815 \cdot 10^{33}$	$1.88982 \cdot 10^5$	$2.63901 \cdot 10^{-3}$	$4.74880 \cdot 10^{-3}$
$(2^{-9}, 2^{-9}, 2^{-9})$	∞	$8.19579 \cdot 10^{12}$	$6.86236 \cdot 10^{-4}$	$1.27756 \cdot 10^{-3}$

conclusions as in example 1.

Example 3 We apply our weighted difference method for the numerical simulation of the model for the dynamics of cells populations in the CFSE proliferation assay ([23]). For $n = k = 1$ we define

$$E = [0, T] \times [x_{min}, x_{max}], \quad E_0 = 0 \times [x_{min}, x_{max}].$$

Consider the one-dimensional hyperbolic partial differential equation with deviated argument

$$\begin{aligned} \partial_t z(t, x) - v(x) \partial_x z(t, x) &= -(\alpha(x) + \beta(x))z(t, x) \\ &\quad + \mathbb{1}_{[x_{min}, x_{max}/\gamma]}(x) 2\gamma \alpha(\gamma x) z(t, \gamma x). \end{aligned} \tag{49}$$

Equation (49) describes the evolution of the cell distribution $z(t, x)$. Cells are structured according to a variable x that denotes the CFSE (carboxyfluorescein succinimidyl ester) expression level. The function v describes the label loss rate and the nonnegative functions α and β represent the proliferation and death rates, respectively. Let the initial CFSE distribution of cells is given by the function

$$z(0, x) = (x - 0.75)^3(1.5 - x)^2(3 - x)^3 \mathbb{1}_{[0.75, 3]} \text{ on } E_0,$$

and we assume the boundary condition $z(t, x_{max}) = 0$, $t > 0$, what means the lack of cells with CFSE intensity above x_{max} for all $t > 0$. \square

In the numerical analysis we put $T = 2$ and $[x_{min}, x_{max}] = [0, 4]$. We take the label dilution factor $\gamma = 2$ and we assume that the natural label loss is proportional to the amount of label $v(x) = 0.11x$. We consider the case with no cellular death ($\beta(x) \equiv 0$) and we take the size-specific division rate function

$$\alpha(x) = (x - 0.25)^2(1 - x)^3 \mathbb{1}_{[0.25, 1]}.$$

Note that the CFL condition has the form $\frac{h_0}{h_1} s v_{max} < 1$, where $v_{max} = \max \{v(x) : x \in [0, 4]\}$.

In order to compare our results we compute the solution at the final time for small step increments: $(h_0, h_1) = (2^{-11}, 2^{-9})$ and we confront this results for numerical solutions calculated on the grids corresponding to the larger steps.

Table 5 contains the maximal errors for different values of s . All methods are stable and produce a good approximation of analytical solution. The best results we obtain for explicit method ($s = 1$).

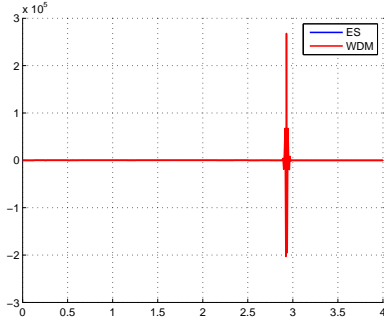
Table 5: Maximal values of errors

(h_0, h_1)	$s = 1$	$s = 0.75$	$s = 0.5$	$s = 0$
$(2^{-7}, 2^{-5})$	$4.4194 \cdot 10^{-2}$	$4.5337 \cdot 10^{-2}$	$4.6472 \cdot 10^{-2}$	$4.8719 \cdot 10^{-2}$
$(2^{-8}, 2^{-6})$	$2.1919 \cdot 10^{-2}$	$2.2560 \cdot 10^{-2}$	$2.3200 \cdot 10^{-2}$	$2.4474 \cdot 10^{-2}$
$(2^{-9}, 2^{-7})$	$9.6783 \cdot 10^{-3}$	$1.0020 \cdot 10^{-2}$	$1.0362 \cdot 10^{-2}$	$1.1044 \cdot 10^{-2}$

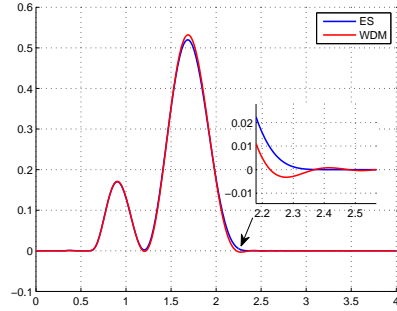
In Table 6 we present results for the step increments (h_0, h_1) which cause violation of condition CFL. The function z reflects density of cells therefore we expect to obtain positive numerical approximation. In the case for $s \in (0.5, 1]$ we loose positivity of function z , even in the situations when we obtain small maximal error, like for $s = 0.75$. Again, we can deduce that the best results are produced by applying the weighted numerical method for $s = 0.5$.

Table 6: Maximal values of errors, violated CFL condition

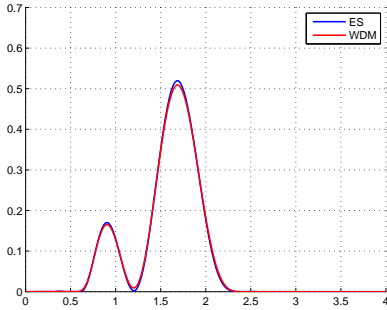
(h_0, h_1)	$s = 1$	$s = 0.75$	$s = 0.5$	$s = 0$
$(2^{-2}, 2^{-6})$	$4.4112 \cdot 10^{-1}$	$2.4512 \cdot 10^{-2}$	$2.3824 \cdot 10^{-2}$	$8.9754 \cdot 10^{-2}$
$(2^{-3}, 2^{-7})$	$2.6794 \cdot 10^5$	$1.3406 \cdot 10^{-2}$	$1.0521 \cdot 10^{-2}$	$4.9951 \cdot 10^{-2}$
$(2^{-4}, 2^{-8})$	$1.6847 \cdot 10^{19}$	$2.0732 \cdot 10^{-2}$	$3.6791 \cdot 10^{-3}$	$2.5244 \cdot 10^{-2}$
$(2^{-5}, 2^{-9})$	$5.0894 \cdot 10^{48}$	$1.0734 \cdot 10^5$	$4.5540 \cdot 10^{-4}$	$1.1490 \cdot 10^{-2}$



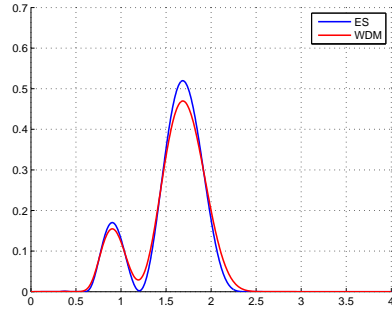
(a) $s = 1$



(b) $s = 0.75$



(c) $s = 0.5$



(d) $s = 0$

Figure 1: Results obtained by weighted numerical method (WDM) for (49) with step increments $(h_0, h_1) = (2^{-3}, 2^{-7})$ at the final time $T = 2$. By ES we denote the solution calculated for a very small steps.

Remark 5 The all examples show that the difference method which we present in the paper is unconditionally stable for $s \in [0, 0.5]$. For $s \in (0.5, 1]$ we need CFL conditions on the mesh (compare with the analysis in [27]). \square

6. Conclusions We considered weighted numerical methods for hyperbolic quasilinear partial differential equations. The complete convergence analysis under suitable assumptions for given functions is presented. We implement our method for two general examples which cover the integral and

deviated differential equations. It is easily seen that the numerical analysis is consistent with theoretical results presented in the paper. The application for numerical solving of mathematical model for the dynamics of cells populations in the CFSE proliferation assay is considered. This example also shows that the weighted numerical method is unconditional stable for the parameter $s \in [0, 0.5]$. Moreover, in the case when the CFL condition is not satisfied, we can observe that numerical solutions obtained by methods for $s \in (0.5, 1]$ brake positivity property which is expected for density function. Application of the methods with $s \in [0, 0.5]$ produce solutions which preserve positivity property. Therefore the analysis of the positivity property of the weighted numerical method for the parameter $s \in [0, 0.5]$ remains open and can be interesting topic of further research, because of the wide applications for the approximation of many models describing biology and epidemiology.

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Ważona metoda różnicowa dla układów quasiliniowych cząstkowych równań różniczkowo-funkcyjnych pierwszego rzędu

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Streszczenie Praca dotyczy zagadnień początkowo brzegowych typu Dirichlet'a dla układów quasiliniowych równań różniczkowo-funkcyjnych. Zamieszczona jest konstrukcja ważonych metod różnicowych dla wyjściowych zagadnień różniczkowych oraz przeprowadzona jest pełna analiza zbieżności. Niezbędne założenia obejmują oszacowania typu Perrona dla funkcji danych względem argumentów funkcyjnych. Dowód stabilności metody różnicowej opiera się na technice porównawczej. Teoretyczne rezultaty zobrazowane są na przykładzie całkowitego równania różniczkowego oraz równań różniczkowych z odchylonym argumentem.

2010 *Klasyfikacja tematyczna AMS (2010)*: 35R10; 65M12; 65M15.

Słowa kluczowe: zagadnienia początkowo brzegowe, metody różnicowe, stabilność i zbieżność, operatory interpolacyjne, oszacowanie błędu, metody porównawcze.



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Communicated by: Henryk Woźniakowski

(Received: 16th of February 2015; revised: 13th of December 2015)