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INFLUENCE OF A VERTEX REMOVING ON THE CONNECTED DOMINATION NUMBER – APPLICATION TO AD-HOC WIRELESS NETWORKS

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ABSTRACT. A minimum connected dominating set (MCDS) can be used as virtual backbone in ad-hoc wireless networks for efficient routing and broadcasting tasks. To find the MCDS is an NP-complete problem even in unit disk graphs. Many suboptimal algorithms are reported in the literature to find the MCDS using local information instead to use global network knowledge, achieving an important reduction in complexity. Since a wireless network continuously changes due to, for example, power restrictions, sensors faults, sensors disconnection etc., it is needed to adapt the MCDS to the new network configuration. In this paper, we study the influence of removing a node on the MCDS and we propose a localized reconfiguration algorithm to obtain the MCDS of the new network topology.

1. INTRODUCTION

An ad-hoc wireless network is a decentralized type of wireless network characterized by a lack of fixed communication infrastructure, so the selection of which nodes forward data is dynamically making by considering the current network connectivity. Several researchers have proposed to use of a virtual backbone in wireless ad-hoc networks as an alternative to the fixed routing infrastructure in classical wired networks [1, 2, 5, 7, 10]. The virtual backbone represents the "skeleton" of the entire network and is used to frequency exchange routing information (traffic conditions, neighbourhood information, etc.) and broadcast a message from one node to all the nodes in the networks.

A *Connected Dominating Set* (CDS) is a subset of nodes such that:

- any two nodes are joined by a path in the network and
- any node in the network either belongs to the CDS (CDS node) or has a neighbour in the CDS (non-CDS node).

The *Minimum CDS* (MCDS) is a natural candidate to be the virtual backbone infrastructure in wireless ad-hoc networks because it guarantees the connectivity of the entire network using the

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minimum number of CDS nodes. To find the MCDS is an NP-complete problem for most graphs and, in general, there exist several MCDS for the same network [3, 6].

Several researchers have proposed fast localized algorithms to find the CDS which running time is constant or polylogarithmic in the network size (see, for instance [4, 7, 9, 12] and references therein). To determine the best method to find the CDS is out of the scope of this paper, but we note that the CDS obtained with some of the referenced methods is not minimum.

In this paper, we assume that the MCDS has been found considering the initial topology of the wireless network. This MCDS can be not efficient when new nodes are activated or current nodes are removed. To compute the MCDS each time a change is detected has a high cost even in network with a reduced number of nodes. It is more efficient to incorporate mechanisms to repair the virtual backbone with local information, although it is possible that the new CDS will be not minimum. The particular case of removing CDS nodes has been considered in [9] but, up to our knowledge, the effect of removing non-CDS nodes has not been considered yet.

In Section 2, we study the effect of deleting a node on the MCDS size. According to our theory, the deletion of a node can give the following consequences: 1) the CDS increases by one or more nodes; 2) the CDS decreases by one node; 3) the CDS size does not change; 4) the CDS is disconnected. The deletion of CDS node can lead to any of these effects, but the non-CDS nodes only can give the effects 2) or 3). In Section 3, we will propose a mechanism to repair the virtual backbone in ad-hoc wireless networks using local information. Finally, Section 4 states some concluding remarks.

2. Graph Theory

We model the network as a *Unit Disk Graph* [3], defined by $G = (V, E)$, where the nodes (vertices) in V are points in the Euclidean plane. We assume that the maximum transmission range is the same for all nodes in the network and it is scaled to one unit. There exists an edge $uv \in E$ if u and v are within the maximum transmission range of each other, i.e., the Euclidean distance is $d_G(u, v) \leq 1$.

For instance, in Figure 1 we can see a network represented by the collection of 5 nodes and the corresponding unit disk graph.

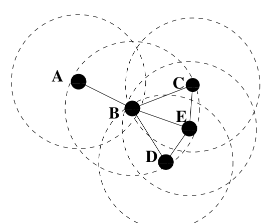

FIGURE 1. A collection of nodes and the corresponding unit disk graph G

2.1. Basic Definitions.

Let $G = (V(G), E(G))$ be a connected graph of order at least three. The *neighbourhood* $N_G(v)$ of a vertex $v \in V(G)$ is the set of all vertices adjacent to v, i.e $N_G(v) = \{u \in V : uv \in E(G)\}$. The *degree* of a vertex v is denoted by $d_G(v) = |N_G(v)|$. For example, for the graph G from Figure 1, $N_G(A) = \{B\}, N_G(C) = \{B, E\}$, so $d_G(A) = 1, d_G(C) = 2$.

An *induced subgraph* of G is a subset of the vertices of a graph G together with any edges whose endpoints are both in this subset. A *cycle graph* is a graph that consists of a single cicle, i.e, some number of vertices connected in a closed chain. The cycle graph with n vertices is denoted by C_n. Every vertex of a cycle graph C_n has degree two. A *complete graph* with n vertices, denoted by K_n, is a graph in which every pair of distinct vertices is connected by a unique edge. A *wheel graph* W_n is a graph with $n \geq 4$ vertices formed by connecting a simple vertex to all vertices of $(n-1)$- cycle.

A subset D of V is *dominating* in G if every vertex of $V - D$ has at least one neighbour in D. A dominating set D is *minimal dominating set* in G if no proper subset $S' \subset S$ is a dominating set of G. A subset D of V is *connected dominating* if D is dominating and the subgraph $G[D]$ induced by D is connected. Let $\gamma(G)$ be the minimum cardinality among all dominating sets in G. The minimum cardinality of a connected dominating set of G is a *connected domination number* of G and is denoted by $\gamma_c(G)$. A minimum connected dominating set of a graph G is called a γ_c-set of G. This parameter was defined by Sampathkumar and Walikar in [11].

For $X \subseteq V$ and $x \in X$, the set $PN_G(x, X) = N_G[x] - N_G[X - \{x\}]$ is called the *private neighbourhood* of x. Note that $x \in PN_G[x, X]$ if and only if x is an isolated vertex in $G[X]$ (the subgraph induced by X in G). It is well-known ([8]) that a dominating set X is minimal if and only if $PN_G(x, X) \neq \emptyset$ for any vertex $x \in X$.

2.2. **Preliminary results.** We will consider removing of a vertex $v \in V$ from the graph G. If the graph $G - v$ is connected, we can classify the vertices according to the influence on the connected domination number of G:

$$V^0(G) = V^0 = \{v \in V : \gamma_c(G) = \gamma_c(G - v)\};$$
$$V^+(G) = V^+ = \{v \in V : \gamma_c(G) < \gamma_c(G - v)\};$$
$$V^-(G) = V^- = \{v \in V : \gamma_c(G) > \gamma_c(G - v)\}.$$

For the case where $G - v$ is disconnected, we define

$$V^r(G) = V^r = \{v \in V : G - v \text{ is disconnected}\}.$$

As a result, we have $V = V^0 \cup V^+ \cup V^- \cup V^r$.

Observation 2.1. For a connected graph G we have:

(1) V^r contains the set of all supports in G, i.e. $S(G) \subseteq V^r$.
(2) If $V^- \neq \emptyset$, then $\gamma_c(G) \geq 2$. In fact, when $\gamma_c(G) = 1$, the number $\gamma_c(G)$ can not decrease and $V = V^0 \cup V^+ \cup V^r$.

Example 2.2.

(1) If G is a cycle $G = C_n$ with $n \geq 4$, then $V = V^-$ (for $n = 3$ is $C_3 = K_3$ and $\gamma_c(C_3) = 1$).
(2) Let G be a wheel W_n with $n \geq 4$ vertices. Since G contains a vertex of degree $n - 1$, $\gamma_c(G) = 1$. If we remove the vertex v of degree $n - 1$ from G, then $G - v$ is a cycle C_{n-1} and $\gamma_c(C_{n-1}) = n - 3$. In this case, removing a vertex v, for $n > 4$ increases the number γ_c and $v \in V^+$.
(3) For a complete graph K_n with $n \geq 2$, we have $V = V^0$.

Lemma 2.3. *Let D be an MCDS of G and $w \in D$. Then the following statements hold:*

(1) $d_G(w) \geq 2$
(2) *if a subgraph $G[D - \{w\}]$ induced by the set $D - \{w\}$ is connected, then w has at least one private neighbour $v \in V - D$.*

Proof.

(1) If $d_G(w) < 2$, then w is an end-vertex. It is easy to observe that any minimum connected dominating set contains no end-vertex. Hence $w \notin D$, a contradiction. We conclude $d_G(w) \geq 2$.

(2) If w has no private neighbour in $V - D$, then $D - \{w\}$ is a dominating set of G. By hypothesis, $G[D - \{w\}]$ is connected. This leads to the contradiction with the minimality of D.

□

The above lemma allows to establish the following conditions to the vertices in V^+.

Proposition 2.4.

(1) *If $w \in V^+$, then w belongs to any minimum connected dominating set of G.*

(2) *Let D be an MCDS of G and $w \in V^+$. If $G[D - \{w\}]$ is connected, then w has at least two private neighbours in $V - D$.*

Proof.

(1) Let D be a minimum connected dominating set of G and $w \notin D$. We have $D \subset V(G - w)$ and D is a connected dominating set in $G - w$. We get $\gamma_c(G - w) \leq |D| = \gamma_c(G)$, which is a contradiction with the fact that $w \in V^+$. Thus, we conclude that $w \in D$.

(2) By 1. above, $w \in D$. By Lemma 2.3, w has at least one private neighbour in $V - D$.

Suppose that w has exactly one private neighbour x, then x is the only vertex which is not dominated in $G - w$. Since $G - w$ is connected, $d_{G-w}(x) \geq 1$. Thus x has a neighbour y in $V - D$. Since D is a dominating set of G, y has a neighbour in D. Thus $(D - \{w\}) \cup \{y\}$ is a connected dominating set of $G - w$ and $\gamma_c(G - w) \leq |D| = \gamma_c(G)$, a contradiction. As a consequence, w has at least two privative neighbours in $V - D$.

□

Observation 2.5. Let D be an MCDS in a graph G.

(1) If $w \in D$ has exactly one neighbour in D, then $G[D - \{w\}]$ is connected.

(2) A node belonging to any MCDS does not have to be in V^+.

The reverse of Proposition 2.4.1. is not necessary true. For instance, Figure 2 shows the graph G with labeled vertex v. It is easy to observe that every MCDS of G contains v, but $\gamma_c(G) = \gamma_c(G - v) = 5$, so $v \notin V^+$.

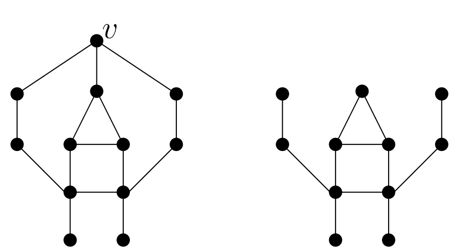

FIGURE 2. Graphs G and $G - v$

2.3. **Theory to CDS Reconfiguration.** Using the graph theory presented above, we will study the case where a vertex v is deleted from the graph. First, we note that for $v \in V^+$ the difference $\gamma_c(G-v)-\gamma_c(G)$ can be arbitrarily large (see, for instance, a wheel from Example 2.2). However, if $v \in V^-$, then it is possible to find a bound for the difference $\gamma_c(G) - \gamma_c(G-v)$.

Theorem 2.6. *Let G be a graph with $n \geq 3$. If $v \in V$ and $G - v$ is connected, then*

$$v \in V^- \text{ if and only if } \gamma_c(G) - \gamma_c(G-v) = 1.$$

Proof. First, we prove that if $v \in V^-$, then $D_v \cap N_G(v) = \emptyset$ for every γ_c- set D_v of $G-v$. In fact, if $N_G(v) \cap D_v \neq \emptyset$, then D_v is also dominating and connected set in G. It implies $\gamma_c(G) \leq \gamma_c(G-v)$, a contradiction with $v \in V^-$.

Now, take $v \in V^-$. So $\gamma_c(G) - \gamma_c(G-v) > 0$. Let D_v be a γ_c-set of $G-v$. We know that $N_G(v) \cap D_v = \emptyset$. Since every vertex of $N_G(v)$ has a neighbour in D_v, we obtain that $D_v \cup \{w\}$ is a connected dominating set in G, where $w \in N_G(v)$. Hence $\gamma_c(G) \leq \gamma_c(G-v)+1$. Finally, from both inequalities we have $\gamma_c(G) - \gamma_c(G-v) = 1$.

The converse is obvious. □

Theorem 2.7. *Let D be an MCDS of G and $v \in V - D$. If $w \in D$ is a neighbour of v such that it has no private neighbour in $V - (D \cup \{v\})$ and $G[D - \{w\}]$ is connected, then:*

(1) *$D - \{w\}$ is an MCDS of $G - v$ and $v \in V^-$;*
(2) *$v \notin D$ and it is the only private neighbour of w in $V - D$;*
(3) *$w \notin V^+$.*

Proof.

(1) First note that $D - \{w\}$ is a dominating set of $G - v$. In fact, we take $x \in G - v$. If $x \notin D$, then there is $z \in D$, $z \neq w$ such that z is a neighbour of x since w has no private neighbour in $V - (D \cup \{v\})$ and D is an MCDS of G. Moreover, by hypothesis, $G[D - \{w\}]$ is connected, so it is an MCDS of $G - v$. Consequently, $\gamma_c(G-v) \leq |D - \{w\}| < \gamma_c(G)$ and we get $v \in V^-$.
(2) By Lemma 2.3, we know that w has some private neighbour in $V - D$ and by the hypothesis w has no private neighbour in $V - (D \cup \{v\})$, then we conclude that v must belong to $V - D$ and it is the only private neighbour of w.
(3) If $w \in V^+$, then by Proposition 2.4, w has at least two privative neighbours in $V - D$. This contradicts statement 2 above.

□

Lemma 2.8. *If $v \in V^-$, then there exists a minimum connected dominating set D such that $v \notin D$.*

Proof. Let $v \in V^-$; then we have $\gamma_c(G-v) < \gamma_c(G)$, for any minimum connected dominating set D_v of $G - v$. As a consequence, v is not dominated by D_v. Of course, $v \notin D_v$. Since G is connected, $N_G(v) \neq \emptyset$; let $x \in N_G(v); x \notin D_v$. Since v is the only vertex not dominated by D_v in G, $D = D_v \cup \{x\}$ is a minimum connected dominating set of G such that $v \notin D$. □

3. Algorithm for Network Reconfiguration

We propose a method to reconfigure the MCDS when a non-MCDS node in the network is removed. The reconfiguration algorithm includes two phases: testing and MCDS updating. Proposition 2.4 guarantees that non-MCDS nodes are not in V^+ and, as a consequence, the reconfigured

MCDS will contain the same or less nodes than the original MCDS, but the size never can grow. In addition, Theorem 2.6 guarantees that, when a node in V^- is removed, MCDS size will be reduced by one node.

We assume that an MCDS has been obtained considering the initial network topology. Nodes in the network are identified by its identification numbers, ID. Each node in the MCDS has two configuration tables with information of its neighbours: the "MCDS table" with the identification number, ID, of neighbours in the MCDS and the "private nodes table" formed by the IDs of the neighbours not connected to other MCDS nodes. Notice that if D is MCDS having at least two vertices and $v \in D$, then every private neighbour of v with respect to D belongs to $V - D$.

Testing phase. The *testing phase* allows to detect removing of the non-MCDS nodes:

- **Step 1::** The node in the MCDS, denoted by w_0, periodically send a REQUEST message to its 1-hop neighbours.
- **Step 2::** When a neighbour receives a REQUEST message, it sends a RESPONSE message.
- **Step 3::** If w_0 does not received the RESPONSE message of the neighbour v, the node w_0 checks the ID of v in the private nodes table:
  - **Step 3.1::** If the ID of v appears in the private nodes table, this ID is removed.
  - **Step 3.2::** If the updated private nodes table is empty, w_0 begins the MCDS updating phase.

Note that the private nodes table of w_0 after removing the node v is empty when v is the only private neighbour of w_0. This situation corresponds to the second consequence of Theorem 2.7. This theorem guarantees that v is a node in V^- and, as a consequence, the MCDS must be updated.

MCDS updating phase. This phase is done when the result of the *testing phase* indicates that the deleted node belongs to V^- and, from Theorem 2.6, we know that the size of the MCDS will be less by one than the original size. In this phase, it is needed to guarantee that the new dominating set is connected, which implies to study if there exists a path between all the neighbours of w_0 (in the dominating set). This task can produce a high overhead in computational cost and messages interchange. For this reason, our algorithm only studies direct paths when w_0 has two neighbours (in the dominating set).

The algorithm is the following:

- **Step 1::** The MCDS node w_0 looks for the "MCDS nodes table":
  - **Case 1::** If the table has only one node, it means that the node w_0 must be eliminated from the MCDS. Go to Step 2.
  - **Case 2::** If the table has two nodes, it is needed to study if there exists a direct connection between the MCDS neighbours of w_0:
    - **Step 2.1::** The node w_0 sends the ASK message with the ID in the "MCDS nodes table" to the other node in this table.
    - **Step 2.2::** The neighbour looks for this ID in its "MCDS nodes table" and answer YES or NO.
    - **Step 2.3::** If the node w_0 receives the answer YES, it determines that the MCDS must be updated and it goes to Step 2. In other case, there is no change in the MCDS.
  - **Case 3::** If the table contains more than two nodes, the MCDS is not updated.

Step 2:: The node w_0 sends an UPDATE-MCDS indidicating that the ID of w_0 must be eliminated from the "MCDS nodes table". Also, if the "MCDS table" of w_0 has only one ID, w_0 sends an UPDATE-PRIVATE message to this ID indicating that the ID of w_0 must be included in "private nodes table".

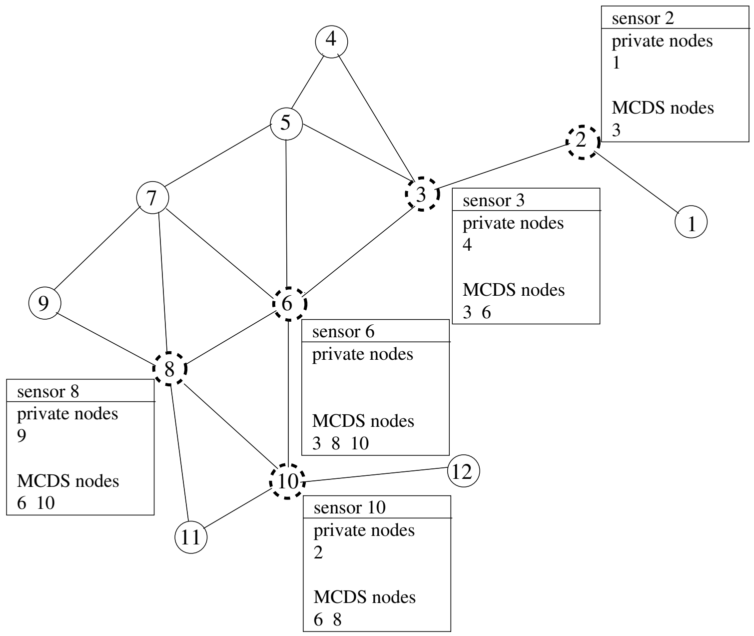

FIGURE 3. Example: Connection between nodes and MCDS

Example 3.1. We will consider the unit disk network in Figure 1 formed by 12 nodes. The connections between the nodes determined by the target regions are presented in Figure 3. There are several MCDS formed by 5 nodes. In particular, we have selected the MCDS formed by nodes $\{2, 3, 6, 8, 10\}$. Each MCDS node has a private nodes table and a MCDS nodes table.

We will explain the effect of disconnecting some non-MCDS nodes on the MCDS.

Disconnection of node 1 in the network of Figure 3. In the *testing phase*, node 2 detects that node 1 has been disconnected and, since the private nodes table is empty after removing the ID of node 1, the *MCDS updating phase* is done. The ”MCDS table” of node 2 has only one node and, using *Case 1*, node 2 is eliminated from the MCDS. As a result, the new MCDS is formed by $\{3, 6, 8, 10\}$. Also, the ID of node 2 is included in the "private node table" of node 3 (see figure 4).

Disconnection of node 4 in the network of Figure 3. In the *testing phase*, node 3 detects that node 4 has been disconnected and removes its ID of the "private nodes table". Since the "private nodes table" of node 3 is empty, the *update MCDS phase* is performed. In this phase, from case 2, node 3 detects that there are not a direct connection between node 2 and 6, and the MCDS is not updated. Figure 5 shows the reconfigured network.

Disconnection of node 9 in the network of Figure 3. In the *testing phase*, node 8 detects that node 9 has been deleted and removes the ID from "private nodes table". Since the "private nodes table" is empty, the *update MCDS phase* is performed. In this phase, from case 2, node 8 detects that there exists a direct connection between node 6 and 10, and the MCDS is updated by removing node 8 from the "MCDS nodes table". The new MCDS is {2,3,6,10} (see figure 6).

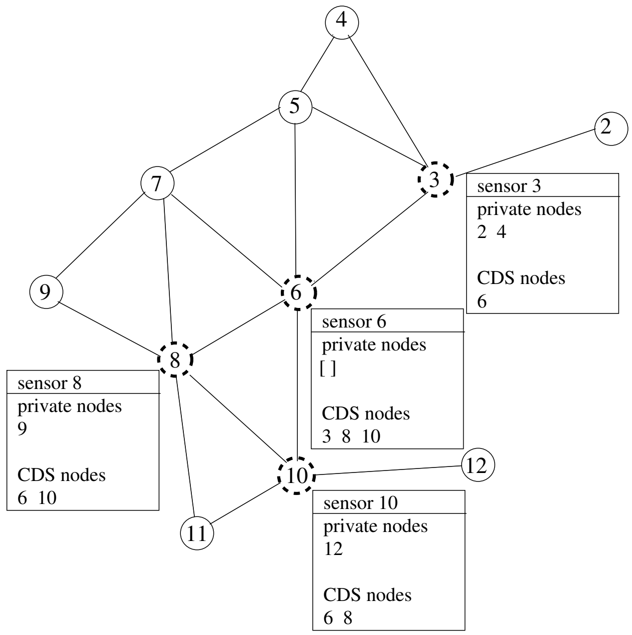

FIGURE 4. Example: Network after deleting of node 1

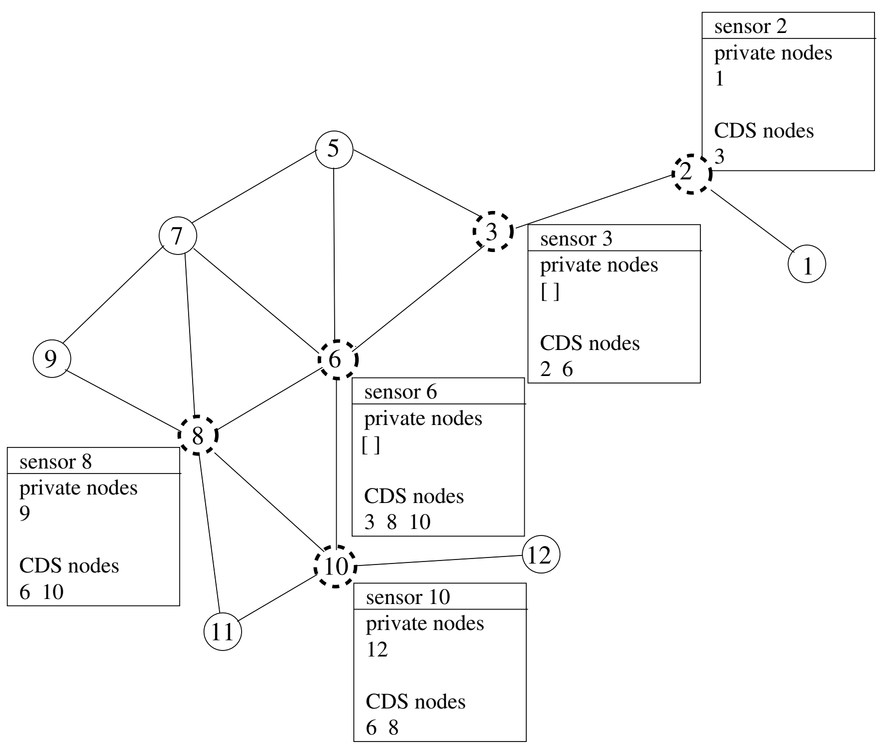

FIGURE 5. Example: Network after deleting of node 4

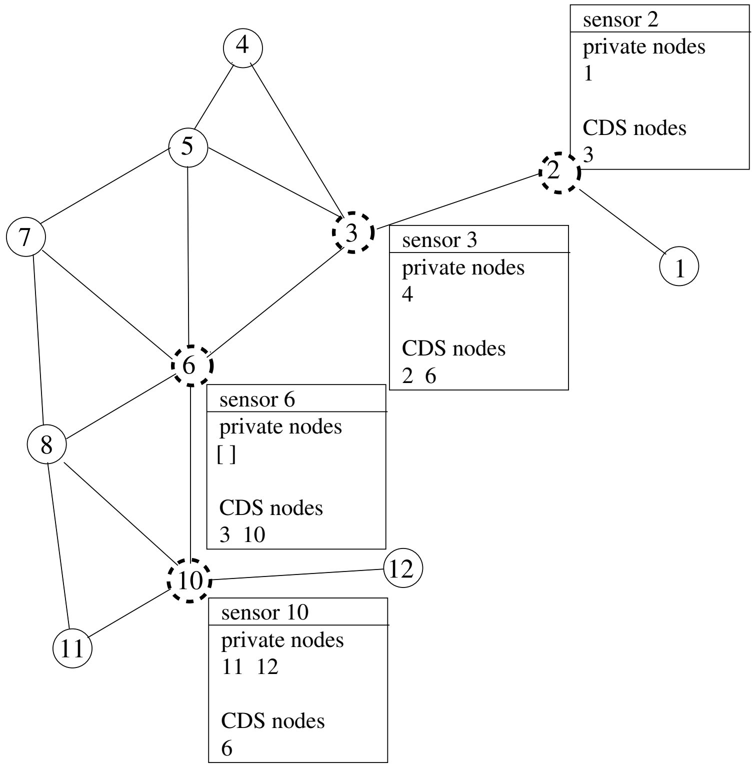

FIGURE 6. Example: Network after deleting of node 9

4. CONCLUSIONS

We presented a graph theory to study the effect of deleting a vertex from a connected graph on the MCDS size. The proposed theory allows to affirm that deleting a non-MCDS node never increases the MCDS size. This result is used to propose an algorithm to reconfigure the backbone of an ad-hoc wireless network using local information, which has a very overhead in computational load and message interchange.

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