ISBN 978-83-942947-6-2
PORTAL CZM 2015

FINITE-TIME LYAPUNOV EXPONENTS IN MODELS OF POPULATION DYNAMICS

KATARZYNA BUSZKO AND KRZYSZTOF STEFAŃSKI

ABSTRACT. In this paper we show how one can adapt Lapunov exponents for studying chaotic transient behavior in nonlinear maps. We characterize this phenomenon introducing average finite-time Lyapunov exponents. As examples for testing the method we take models of population dynamics that can generate both asymptotic chaos and transient chaos inside periodic windows.

1. INTRODUCTION

Lyapunov exponents (LEs) belong to the best known instruments used to investigate nonlinear dynamical systems. Majority of literature on such systems is focused mainly on their asymptotic behavior that, depending on values of control parameters, can be either chaotic or periodic (in general – regular). According to definition of LEs, the largest (in the case of discrete-time systems) or the largest nonvanishing of them (in the case of continues-time systems) tells about the asymptotic behavior of the analyzed dynamical system.

In models of biological or medical systems special attention is often payed to asymptotically periodic behavior. It is well known, however, that such a behavior, in the case of systems from periodic windows in bifurcation diagram, is preceded by chaotic transient behavior. Transient chaos, although less exploited than its asymptotic counterpart, was discussed in quite few publications (cf. [1–3]). In this paper we show, that after some modifications it is possible to adapt LEs for description of such behavior. Referring to definition of LE, we introduce the notion of average finite-time Lyapunov Exponents (AFTLE) and use it to detect and estimate chaotic transient behavior inside periodic windows.

The paper is arranged as follows: in Section 1 definition of LE is briefly recalled and definition of the average finite-time Lyapunov exponents (AFTLE) is introduced. In Section 2 numerical estimates of AFTLE for the family of 1-dimensional logistic model of population dynamics are presented and discussed. In Section 3 similar estimates for the family of 2-dimensional coupled logistic maps are discussed.

Publication co-financed by the European Union as part of the European Social Fund within the project Center for Applications of Mathematics

2. LYAPUNOV EXPONENTS

Lyapunov exponents characterize the average rate of growth of infinitesimal initial perturbations of a state of a system. To recall definition of Lyapunov exponents for dynamical systems with discrete time, let us consider the system (X, π) given by

$$\pi(n, x) = f^n(x) \quad \text{for} \quad n = 0, 1, \ldots, \tag{2.1}$$

where $X \subset \mathbb{R}^d$ and $f : X \longrightarrow X$ is a smooth function. If X is 1-dimensional one obtains only one LE expressed by the formula (see, e.g., [8]):

$$\lambda(x_0) = \lim_{n\to\infty} \frac{1}{n} \sum_{s=0}^{n-1} \ln |f'(x_s)|, \quad \text{where} \quad x_s = f^s(x_0), \tag{2.2}$$

In the case of 2-dimensional X, when $f = (f_1, f_2)$, there appear two Lyapunov exponents defined as:

$$\lambda_i(x_0) = \lim_{n\to\infty} \frac{1}{n} \ln \|\chi_i(n)\| \quad i = 1, 2, \tag{2.3}$$

where $\chi_i(n)$ are eigenvalues of the matrix obtained by multiplying Jacobian matrices obtained in each of n iterations. Although LE formally depends on the initial state x_0, in practice it does not since it follows from the Multiplicative Ergodic Theorem [10], that λ_i for fixed i and for almost every $x_0 \in X$ (with respect to the ergodic measure μ) has constant value.

It follows from the definition of LE that a systems is chaotic, when it has at least one positive Lyapunov exponent and generates trajectories confined inside a compact subset of X. In the case of a discrete-time system negative signs of all LEs indicate an attracting periodic orbit. In practice estimates of LEs for discrete-time systems can be obtained numerically by implementing Formulae (2.2) or (2.3). LEs are defined for $t \longrightarrow \infty$, which means, that they are useful solely in the case of description of the asymptotic behavior of dynamical systems. In reality, however, one cannot observe infinitely long time series and quite often only transient behavior can be noticed. On the other hand, such kind of behaviors like chaotic transients observed in dynamical systems disappear when $t \longrightarrow \infty$. To be able to detect them, we introduce AFTLE $\overline{\lambda}_n$, referring to the definition (2.2).

At first we define finite-time estimate of Lyapunov exponent (FTLE) for 1-dimensional maps

$$\lambda_n(x_0) = \frac{1}{n} \sum_{i=0}^{n-1} \ln|f'(x_i)| \tag{2.4}$$

Values of $\lambda_n(x_0)$ depend strongly on the initial condition x_0, therefore we introduce average finite-time Lyapunov exponent $\overline{\lambda}_n$:

$$\overline{\lambda}_n = \frac{\sum_{j=1}^{J} \lambda_n(s_j)}{m}, \tag{2.5}$$

where $s_j = x_0(j)$ is the initial point of the jth of J trajectories over which the FTLEs are averaged in the formula (2.5). It is clear that the estimate may depend on the distribution of initial points s_j but, contrary to the case of FTLE, where even a slight change of x_0 can result in an essential change of FTLE's value, a slight change of distribution of initial points would change value of AFTLE only marginally.

In the case of 1-dimensional chaotic map apart from the AFTLE one can easily define density based LE (DBLE):

$$\lambda^{\star}_{\rho} = \int_{-\infty}^{\infty} \log |f'(x)| \rho(x) \mathrm{d}x, \tag{2.6}$$

where ρ is a distribution density of initial points. It is obvious that the estimate depends on the density ρ although weekly (in the same sense, the AFTLE does). In particular one should expect that if ρ is the density of invariant measure DBLE $\lambda^{\star}$ coincides with LE λ.

3. Average finite-time Lyapunov exponents in one-dimensional model of population dynamics

The best known and very simple dynamical system that can generate chaotic evolution is defined by noninvertible logistic map that can describe evolution of an isolated, homogenous population. The logistic model of population growth is given by the formula:

$$x_{n+1} = f(x_n, r) = r x_n (1 - x_n), \tag{3.1}$$

where $x_n, x_{n+1} \in [0, 1]$, $n \in \mathbb{N}$ and $r \in [0, 4]$ is control parameter of the family of logistic maps.

The model describes seasonal reproduction of a species. Individuals of the species live in an isolated habitat and do not interact with other populations. Parameter r denotes the average number of offsprings per specimen. In this model fertility's reduction connected with competition for food is taken into account by nonlinearity [4–6]. Character of dynamics of the population depends on a particular value of the control parameter r.For some values of r one can observe chaotic evolution while for others the asymptotic dynamics can be periodic. In Fig.1a bifurcation diagram of the family of logistic maps is shown. An ideal bifurcation diagram would be a plot of sets of accumulation points of typical trajectories versus the control parameter r thus representing asymptotic properties of the family of maps. An approximation to such an ideal is created by plotting points of trajectories with omission of their initial sections. The diagram from Fig. 1a shows properties of logistic maps for $r \in [1, 4]$. In Fig. 1b LE for the same family is plotted for the same range of r. As one can see, chaotic behavior corresponds to positive values of LE while negative values of LE appear for such values of r for which logistic maps generate attracting periodic orbits, including those inside periodic windows riving the bifurcation diagram.

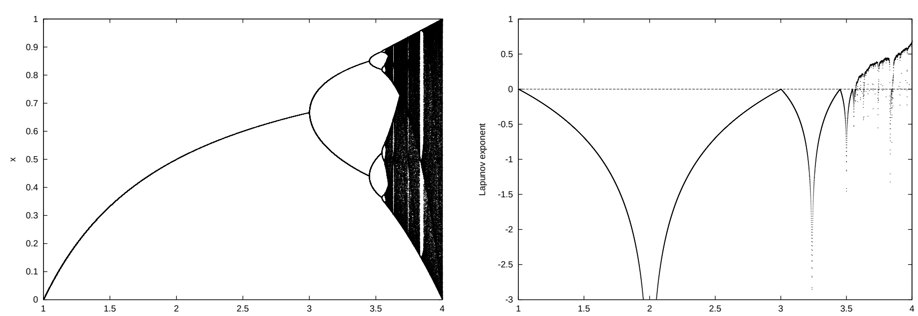

FIGURE 1. The family of logistic maps for $r \in [1, 4]$: a) bifurcation diagram, b) diagram of LE

We are interested in the evolution of the system inside periodic windows, where the asymptotic evolution of trajectories is periodic but their initial sections (sometimes quite long) typically exhibit chaotic features. It means, that asymptotically-periodic behavior is preceded by transient chaotic behavior. It can be illustrated by a slight modification of the procedure of generating bifurcation diagram, by plotting also initial sections of trajectories. In Fig. 2a such a modified bifurcation diagram for $r \in [3.9901, 3.9905]$ and $n = 100$ initial points is shown. One observes features of chaotic evolution for the whole range of r. In Fig. 2b such a 'bifurcation diagram' for the same range of r and $n = 1000$ initial points is shown. In this case a periodic window vaguely emerges. It can be exposed more explicatively by creating a standard bifurcation diagram. Such a diagram is shown in Fig. 2c. It is created of the final 200 points of sections of trajectories $n = 10000$ iterates long. Now the window of period 5, vaguely visible in Fig. 2b, is pronounced.

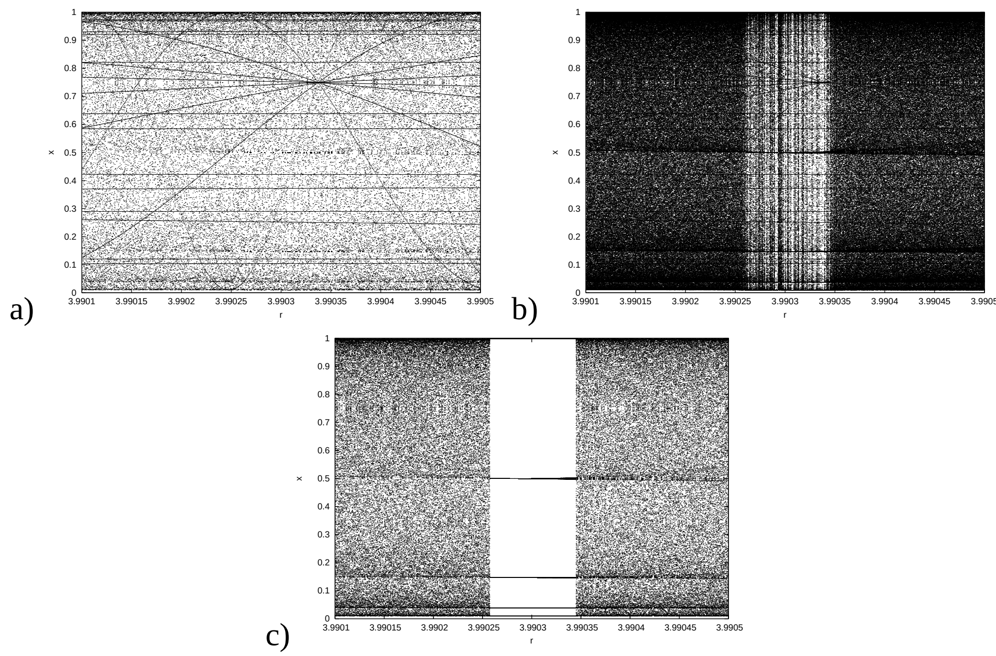

FIGURE 2. Bifurcation diagram of the family of logistic maps for $r \in [3.9901, 3.9905]$ plotted for a) 100 initial iterates, b) 1000 initial iterates c) 200 final of 10000 iterates

To illustrate transient chaos we plotted two typical time series generated by the logistic map (3.1) for $r = 3.9903$ with initial points: $x_0(1) = 0.2$ and $x_0(2) = 0.89$. In both cases the initial evolution of trajectories exhibits chaotic features but after $n \approx 100$ (Fig.3a) in the case of the first trajectory and after $n \approx 300$ (Fig.3b) in the case of the second one the evolution becomes periodic.

LEs computed with (2.2) for the mentioned initial points are: $\lambda(x_0(1)) = -0.0499$ and $\lambda(x_0(2)) = -0.0499$, which is consistent with the statement on independence of LE of the initial state. These values of LE confirm, that for the analyzed value of r the asymptotic behavior is periodic. Computations for $n = 100$ give values of FTLEs $\lambda_{100}(x_0(1)) = 0.179952$, and $\lambda_{100}(x_0(2)) = 0.079998$ which indicates chaotic evolution, and shows strong dependence of FTLE on the initial state. The analyzed chaotic behavior has finite duration but for sufficiently small number of iterations it is impossible to distinguish such transient chaotic behavior from the asymptotically chaotic one. We name chaotic transient behavior 'rambling' and, consequently, its duration is called 'rambling

a) b)

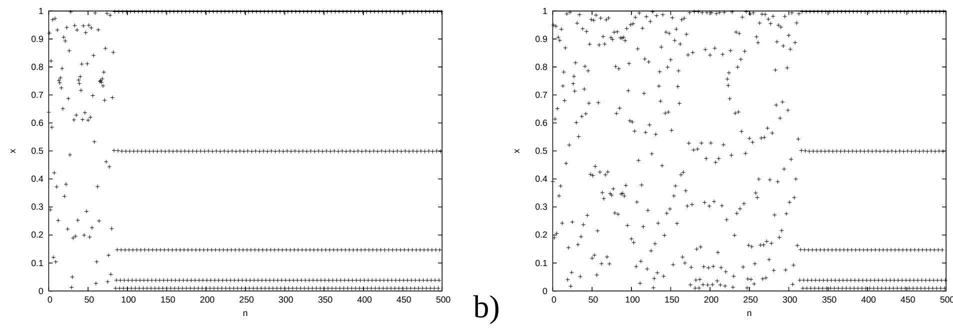

FIGURE 3. Time series generated by the logistic map for $r = 3.9903$ inside the window $(5, 3)$ for $n = 500$ iterates from the initial points: a)$s_1 = x_0(1) = 0.2$ b)$s_2 = x_0(2) = 0.89$

time'. For a single trajectory the rambling time is the number of points, that do not approach to the periodic attractor A. The formal definition of the rambling time is based on the surrounding area of the periodic attractor A within which any trajectory uniformly converges to the attractor. The number of trajectory points outside the area is the rambling time. In our previous papers two methods of determining rambling time, the black intervals method and the distance method [2, 3] were presented.

Now we will show that the introduced AFTLE (2.5) characterizes transient chaos correctly.

The first problem to be solved, however, consists in checking how much values of AFTLE computed according to Eq. (2.5) depend on the distribution of initial points. One can compute AFTLE with initial points s_i uniformly distributed on the interval $[0, 1]$. Another natural choice would be the distribution according to the invariant density ρ^{inv} outside periodic windows and according to the quasi-invariant density ρ^{qinv} inside a window. The invariant measure density ρ^{inv} is a fixed point of the Frobenius-Peron equation [2]:

$$\rho_{n+1}(x) = \widehat{F}\rho_n(x) = \int_0^1 \delta[x - f(y)]\rho_n(y)dy \tag{3.2}$$

The invariant density in the case of asymptotically chaotic evolution is absolutely continuous, while in the case of asymptotically periodic evolution it is a distribution. Unfortunately, in majority of even quite simple cases, except for the case of invariant distribution for a system generating periodic attractor, neither subsequent iterates ρ_n of the Frobenius-Peron equation nor its absolutely continuous fixed point ρ^{inv} can be found in analytical form. Therefore, to learn about their shape one is forced to construct their approximation using histograms. We have constructed such histograms, dividing the interval $[0, 1]$ into L equal subintervals I_l. The height of the lth column of the histogram approximating density ρ_n is given by Lj_l/J where j_l denotes number of points of any of J trajectories that fall into the subinterval I_l in the nth iteration. Such histograms for the windows $(5, 1)$ and $(5, 3)$, obtained from the uniform initial distribution after $n = 20$ iterates are shown in Fig. 4. As is visible, the distribution of the points is not uniform and asymptotically it would consist of 5 columns. As comprehensive studies have shown, for intermediate number of iterations three components: the ephemeric ρ^{eph}, the quasi-invariant ρ^{qinv} and the asymptotic ρ^{asp} can be distinguished in the density ρ [2]. The ephemeric component ρ^{eph} vanishes rapidly, the quasi-invariant component ρ^{qinv} decreases geometrically with n, keeping its shape practically unchanged, and the asymptotic component ρ^{asp} increases with n and has 5 accumulation points in the form of linear combinations of 5 Dirac deltas.

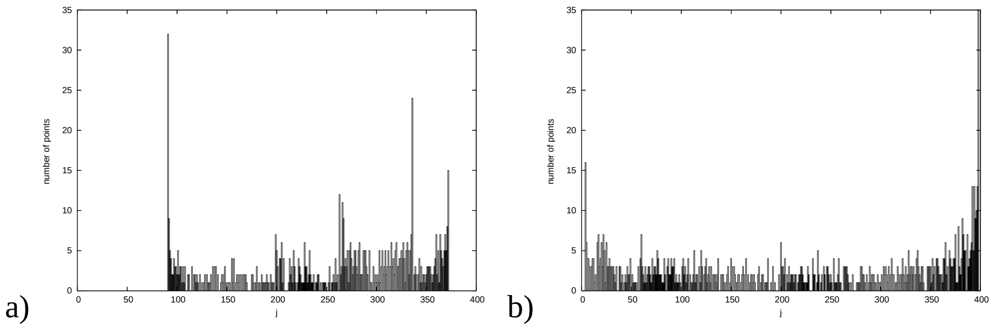

FIGURE 4. Histograms for distribution of points for a set of trajectories for: a) $r = 3.74$ inside the window $(5,1)$, b) $r = 3.99029$ inside the window $(5,3)$.

In our numerical tests AFTLE (2.5) have been computed for initial points s_i distributed uniformly on the $[0,1]$ interval and according to the histograms obtained after $n = 10^4$ iterations for $J = 10^3$ trajectories with initial states distributed uniformly on the $[0,1]$ interval. We also computed DBLEs according to Eq. (2.6) for the uniform density on the [0,1] interval and for the histogram approximating ρ^{inv} outside a window and ρ^{qinv} inside it. Numerical values of all four parameters for three values of the control parameter r near the window (5,3) and for three values inside it are presented in Fig 5. Values of LE (2.6) are also shown for comparison.

One can notice that values of DBLE (2.6) for the uniform density has practically the same magnitude for all tested values of r, both outside and inside the window. Values of DBLE for the histograms approximating ρ^{inv} or ρ^{qinv} are slightly larger and practically coincide with values of LE, AFTLE while outside the window, and inside the window they are slightly smaller than the values of DBLE for uniform density but remain positive. On the other hand, values of LE and of AFTLE both for uniform and quasi-invariant distribution of initial points are almost identical also inside the window. Although with $n = 10^4$ values of AFTLE for the uniform and quasi-invariant distribution barely differ, the latter one gives a bit better approximation to LE.

In Fig. 6 one can see plots of AFTLE $\overline{\lambda}_n$ versus r for all three windows of period 5 and for various numbers of iterations n. Fig. 6a shows the AFTLEs: $\overline{\lambda}_{10}$, $\overline{\lambda}_{30}$, and $\overline{\lambda}_{40}$ for the window $(5,1)$, which is the widest window of period 5. As is clear, negative values of AFTLE appear only for $n \approx 40$ iterations for values of control parameter r close to r_{sst} for which the superstable cycle occurs. In Fig. 6b the AFTLEs: $\overline{\lambda}_{25}$, $\overline{\lambda}_{50}$, and $\overline{\lambda}_{100}$ for the window (5,2) are plotted. In this case only $\overline{\lambda}_{100}$ has negative values for values of $r \approx r_{\text{sst}}$. It means that for $n \approx 100$ and r close to r_{sst} chaotic transient behavior is statistically over although some rambling trajectories may still remain. In Fig. 6c similar plots of $\overline{\lambda}_n$ for the window $(5,3)$, which is the narrowest window of period 5, and for $n = 100$, $n = 300$, $n = 500$ are shown. In this case only $\overline{\lambda}_{500}$ has negative values.

The above results suggest a connection between the rambling time $\overline{M}_{\text{rb}}(r_{\text{sst}})$ and the value of $n = n_{\text{neg}}$ for which negative values of $\overline{\lambda}_{n_{\text{neg}}}$ appear in periodic window for the first time. We have determined the rambling time $\overline{M}_{\text{rb}}(r_{sst})$ [2,3] and values of n_{neg} for all periodic windows of periods from 3 to 8. The results are shown in Table 1.

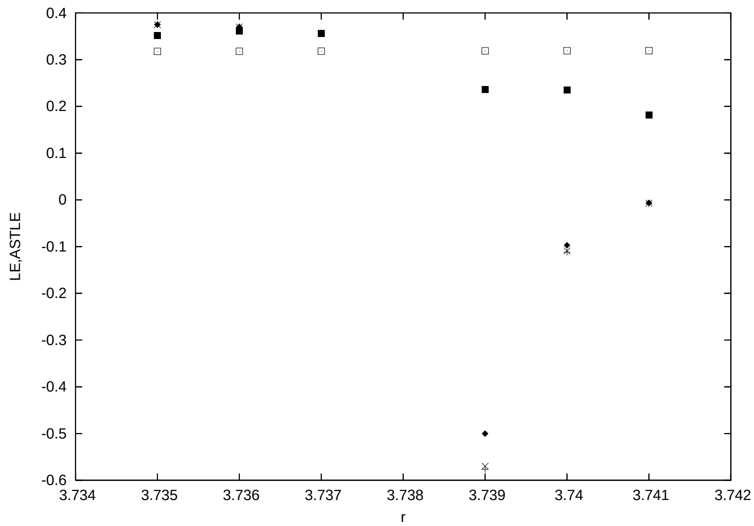

FIGURE 5. Values of Lyapunov exponents in the vicinity (3 points from the left) and inside (3 points from the right) the window $(5,1)$: LE (2.1)-$+$, AFTLE (2.5) with uniformly distributed initial points on the interval $[0,1]$ – $\times$, AFLTE (2.5) with initial points distributed according to the measure ρ – ⧫,. . . with uniformly distributed initial points on the interval $[0,1]$-□,. . . (2.6) with initial points uniformly distributed on the interval $[0,1]$,. . . (2.6) with initial points distributed according to the measure ρ-■.

Apparently $n_{\rm neg}$ is a measure of duration of transient chaos and its values shown in Table 1 agree with the rule revealed in our earlier papers that in the case of the family of logistic maps the narrower the window, the longer the transient chaos duration [2,3]. Values of the ratio $\overline{M}_{\rm rb}(r_{\rm sst})/n_{\rm neg}$ that vary from 0.5 to 3.0 shown in Table 1, however, force one to admit that $n_{\rm neg}$ is a less precise measure of duration of transient chaos than rambling time $\overline{M}_{\rm rb}$. Nevertheless it is clear that the pace at which AFTLE inside a periodic windows decreases is closely connected with rambling time for maps inside it. A model describing this process quantitatively is presented in [11].

4. AVERAGE FINITE-TIME LYAPUNOV EXPONENTS IN 2-D MODEL OF POPULATION DYNAMICS

Such investigations can be extended onto the 2-dimensional logistic model of population dynamics, given by the formula:

$$
\begin{aligned}
x_{n+1} &= dx_n r_x(1-x_n) + (1-d)y_n r_y(1-y_n), \\
y_{n+1} &= (1-d)x_n r_x(1-x_n) + dy_n r_y(1-y_n),
\end{aligned} \tag{4.1}
$$

where $x_n, x_{n+1}, y_n, y_{n+1} \in [0,1]$, n is natural number, $r_x, r_y > 0$ and $d \in [0,1]$. The model describes two colonies of a species, that interact by mutual migration. The migration of individuals may change population dynamics in both colonies. Parameters r_x and r_y represent growth rates in the colonies and the isolation parameter d represents coupling between their dynamics. The coupling is strongest for $d = 0$ while for $d = 1$ the colonies do not interact. The simplest, symmetric case $r_x = r_y$ was discussed previously [4]. In Fig. 7a bifurcation diagram with respect

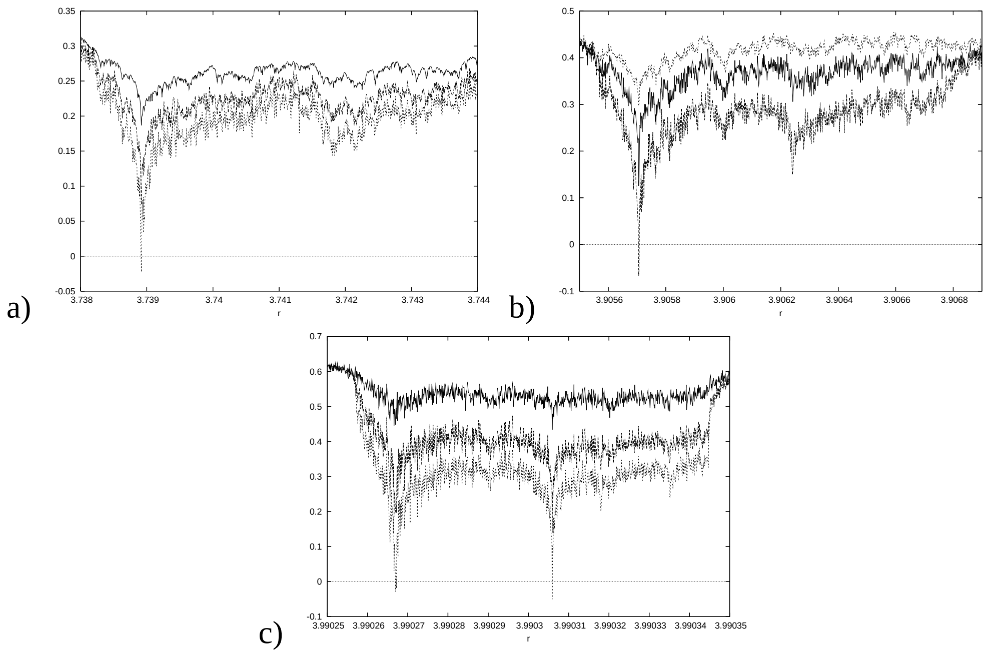

FIGURE 6. Plots of AFTLE $\overline{\lambda}_n$ inside period 5 windows: a) $(5,1)$ – from the top to the bottom – for $n = 10$, $n = 30$, $n = 40$, b) $(5,2)$ for $n = 25$, $n = 50$, and $n = 100$, c) $(5,3)$ for $n = 100$, $n = 300$, and $n = 500$

to d is plotted for $r_x = r_y = 4$. Our attention is focused on the two periodic windows visible in the diagram. In Fig. 7b values of AFTLE: $\overline{\lambda}_{20}$, $\overline{\lambda}_{40}$, $\overline{\lambda}_{50}$, $\overline{\lambda}_{100}$, and $\overline{\lambda}_{200}$ are plotted.

Outside periodic windows, as expected, values of AFTLE $\overline{\lambda}_n$ do not depend on n. Inside the periodic windows they decrease with n, and become negative only for $n = 200$. For $n < 200$ transient chaotic behavior prevails.

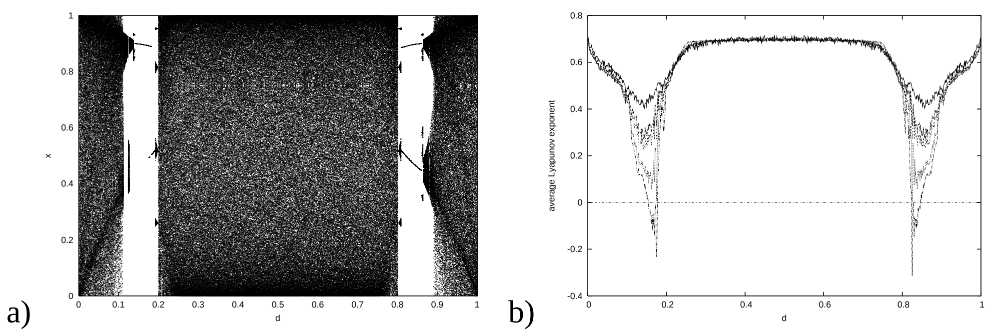

FIGURE 7. a) Bifurcation diagram of the family (4.1) for $r_x = r_y = 4.0$, $d \in [0,1]$, b) Values of AFTLE $\overline{\lambda}_n$: from the top to the bottom – for $n = 20$, $n = 40$, $n = 50$, $n = 100$, and $n = 200$

An interesting phenomena can be observed after breaking symmetry between the colonies ($r_x \neq r_y$). This can be seen in Fig. 8a where bifurcation diagram for $r_x = 3.5$, $r_y = 4$, and $d \in [0,1]$ is shown. Two wide periodic windows of period 5 are visible in the diagram: one for $d \in [0.2, 0.3]$

TABLE 1. Values of rambling time $\overline{M}_{\rm rb}(r_{\rm sst})$ and $n_{\rm neg}$ for windows of periods $k = 3, \ldots, 8$

window's number	r_{sst}	n_{neg}	$\overline{M}_{\rm rb}$	$\overline{M}_{\bf rb}/n_{\rm neg}$
(3,1)	3.831874	15	25	1.6
(4,1)	3.960270	56	118	2.1
(5,1)	3.738915	32	53	1.6
(5,2)	3.905706	58	119	2.05
(5,3)	3.990267	169	469	2.77
(6,1)	3.627557	29	45	1.55
(6,2)	3.937536	162	375	2.3
(6,3)	3.977766	283	616	2.17
(6,4)	3.997583	1106	2089	1.8
(7,1)	3.701769	71	147	2.0
(7,2)	3.774214	77	180	2.3
(7,3)	3.886046	146	349	2.3
(7,4)	3.922193	198	585	2.95
(7,5)	3.951032	341	680	1.99
(7,6)	3.968977	343	726	2.1
(7,7)	3.984747	685	1307	1.9
(7,8)	3.994538	1183	2131	1.8
(7,9)	3.999397	4348	6155	1.41
(8,1)	3.662192	90	221	2.45
(8,2)	3.800771	183	311	1.6
(8,3)	3.870541	262	448	1.7
(8,4)	3.899469	384	531	1.3
(8,5)	3.912047	381	597	1.5
(8,6)	3.930473	886	1173	1.32
(8,7)	3.944213	1012	1334	1.3
(8,8)	3.973724	1816	2044	1.12
(8,9)	3.9814099	2619	2334	0.8
(8,10)	3.987745	2935	3359	1.14
(8,11)	3.992519	4408	4725	1.07
(8,12)	3.996219	4948	6162	1.24
(8,13)	3.998642	12978	10854	1.19
(8,14)	3.999849	61713	30649	0.49

and the second for $d \in [0.32, 0.4]$. We have estimated the AFTLEs: $\overline{\lambda}_{30}$, $\overline{\lambda}_{50}$, $\overline{\lambda}_{100}$, $\overline{\lambda}_{200}$, $\overline{\lambda}_{500}$, and $\overline{\lambda}_{700}$ inside these windows, using formula (2.5). Their plots for $d \in [0.21, 0.41]$ are shown in the Fig. 8b. Positive values of $\overline{\lambda}_n$ indicate chaotic transient behavior inside periodic windows. Negative values of $\overline{\lambda}_n$ for the first window occur only for $n_{\rm neg} \approx 700$ while for the second window it happens already for $n \approx 200$. It would suggest that in the second window rambling time should be shorter than that in the first window.

In Figs. 8 b and d bifurcation diagram and AFTLEs are shown for $r_x = 3.8$, $r_y = 2.0$ and $d \in [0.928, 0.9325]$. A periodic window of period 6 as well as signatures of transient chaos inside of it are clearly visible.

Negative values of AFTLE in this case occur for the first time for $n \approx 50$.

We determined rambling time $\overline{M}_{\rm rbd}$ for some periodic windows of the analyzed family of maps using the distance method [3]. Values of $\overline{M}_{\rm rbd}$ and values of $n_{\rm neg}$ are given in Table 2 as was done for the 1-dimensional family of maps.

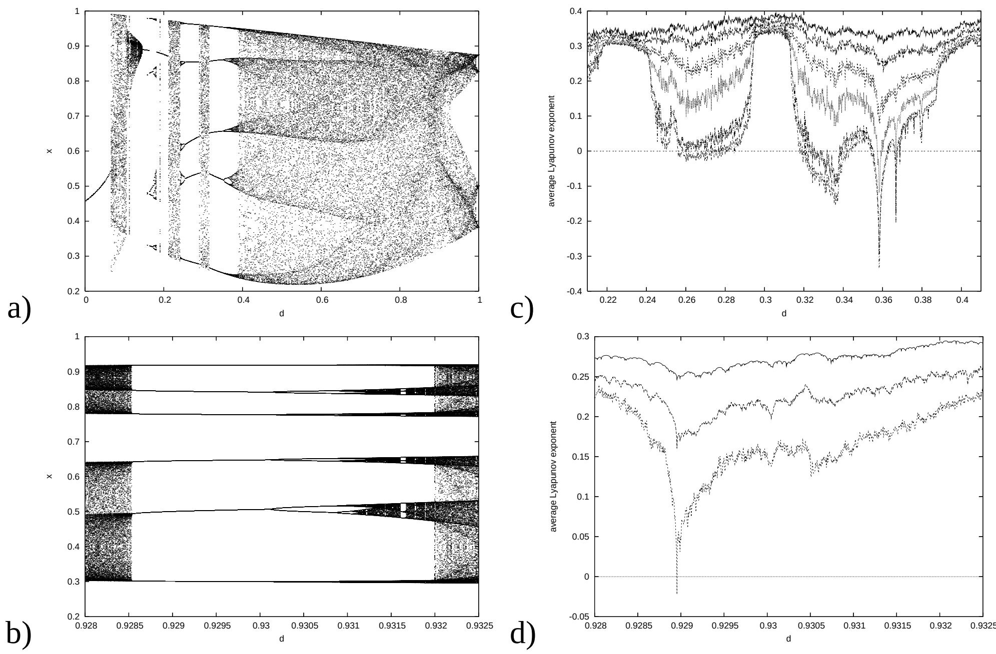

FIGURE 8. Bifurcation diagram for: a) $r_x = 3.5$, $r_y = 4.0$, $d \in [0, 1]$, b) $r_x = 3.8$, $r_y = 2.0$, $d \in [0.928, 0.9325]$. Values of AFTLE $\overline{\lambda_n}$ for: c) $r_x = 3.5$, $r_y = 4.0$, $d \in [0.21, 0.41]$ and for $n = 30$, $n = 50$, $n = 100$, $n = 200$, $n = 500$, $n = 700$, d)$r_x = 3.8$, $r_y = 2.0$, $d \in [0.928, 0.9325]$ and from the top: $n = 20$, $n = 30$, $n = 50$

TABLE 2. Values of $\overline{M}_{\rm rbd}$ and $n_{\rm neg}$ for a numbers of windows in the 2-dimensional population model for the values of control parameter $d = d_{ms}$, that indicates the most stable cycle

r_x	r_y	$d_{\rm ms}$	window's number	$n_{\rm neg}$	$\overline{M}_{\rm rbd}$	$\overline{M}_{\rm rbd}/n_{\rm neg}$
3.8	2	0.97489	5 (5)	138	91.4	0.66
3.8	2	0.95966	7 (4)	224	183.14	0.82
3.8	2	0.929	6 (3)	63	90.4	1.43
3.8	2	0.919734	10 (2)	165	144.6	0.88
3.8	2	0.9100699	12 (1)	141	167.8	1.19
2.9	3.9	0.97079	3 (5)	91	83.29	0.92
2.9	3.9	0.92554	5 (4)	845	546.45	0.65
2.9	3.9	0.871994	8 (3)	1230	768.84	0.63
2.9	3.9	0.846779	6 (2)	106	112.6	1.06
2.9	3.9	0.827949	10 (1)	198	158.4	0.80

In this case the ratio $\overline{M}_{\rm rbd}/n_{\rm neg}$ varies from 0.63 to 1.43 and it shows again that values of $n_{\rm neg}$ would be in general a poor quantitative measure of rambling time inside the windows, although they indicate close interdependence between the rambling time $\overline{M}_{\rm rbd}$ and time dependence of values of AFTLE. Modeling of quantitative dependence in the case of families of 2-dimensional maps, however, will be much more complicated than in it is in the case of families of 1-dimensional maps [11].

The presented numerical results show clearly that periodic evolution in population dynamics is preceded by chaotic transient evolution. In consequence, even if the control parameters of the system r (in the 1-dimensional family of maps), r_x, r_y and d (in the 2-dimensional family of maps) correspond to asymptotically periodic evolution, one observes chaotic behavior for a shorter or longer time in the case of majority of initial states of the system. Transient chaos has a finite duration but sometimes it is prolonged, and finite time of observation may be too short to see transition to periodic behavior.

5. Final remarks

We have shown that although Lyapunov exponents characterize asymptotic behavior of dynamical systems their short-time estimates can be used to characterize chaotic transient evolution. As an example we have taken two models of population dynamics. The AFTLE's dependence on the number of iterations can reveal chaotic transient behavior that plays significant role in the discussed models. What's more, value of n_{neg} allows one to estimate, although rather roughly, duration of chaotic transient behavior without necessity of establishing any criterion of the end of rambling for an individual trajectory that is necessary both in the case of the black intervals method [1, 3] and in the distance method [11].

References

[1] K.Buszko,K.Stefański,*Transient Dynamics Inside Periodic Windows and in Their Vicinity I.Logistic Maps*, Open Sys. and Information Dyn. 10 (2003), 183–203.
[2] K.Buszko, K.Stefański, *Measuring transient chaos in nonlinear one- and two-dimensional maps*, Chaos, Solitons and Fractals 27 (2006), 630–646.
[3] B.Hasselblatt, A. Katok, *A first Course in Dynamics*, Cambrige University Press 2003.
[4] J.Jacobs, E.Ott, B.R. Hunt, *Scaling of the durations of chaotic transients in windows of attracting periodicity*, Phys. Rev.E 56 (1997), 6508.
[5] E. Ott, *Chaos in Dynamical Systems*, Cambrige University Press (1994).
[6] V. I. Oseledec,*A Multiplicative Ergodic Theorem*, Trans. Moscow Math. Soc. 19 (1968), 197–231.
[7] K.Piecyk, K.Buszko, *Lyapunov Exponents in Biomedical Sciences*, Proceedings of the Eleventh National Conference Application of Mathematics in Biology and Medicine, Zawoja 2005, 43–48.
[8] H.G. Schuster, *Deterministic Chaos*, VCH, Weinheim (1988).
[9] K.Stefański, K.Buszko, K.Piecyk, *Transient chaos measurements using finite-time Lyapunov exponents*, Chaos 20 (2010), 033117-1–033117-13.
[10] F. E. Uwadia, N. Raju, *Dynamics of Coupled Nonlinear Maps and Its Application to Ecological Modelling*, Applied Mathematics and Computation 82 (1997), 137–179.
[11] P. Wolf, J.B. Swift, H.L. Swinney, J.A. Vastano, *Determining Lyapunov exponents from a time series*, Physica D (1984), 285–317.

Katarzyna Buszko
Department of Theoretical Foundations of Biomedical Sciences,& Medical Informatics, Nicolaus Copernicus University in Toruń, Collegium Medicum in Bydgoszcz, ul. Jagiellońska 13, 85-067 Bydgoszcz, Poland
E-mail address: buszko@cm.umk.pl

Krzysztof Stefański
Department of Theoretical Foundations of Biomedical Sciences,& Medical Informatics, Nicolaus Copernicus University in Toruń, Collegium Medicum in Bydgoszcz, ul. Jagiellońska 13, 85-067 Bydgoszcz, Poland
E-mail address: stefan@fizyka.umk.pl