ISBN 978-83-942807-8-9
PORTAL CZM 2015

A NEW APPROACH FOR USING CARDINAL B–SPLINES

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ABSTRACT. In this paper we propose a new approach for using cardinal B-splines in numerical methods. The concept is based on shifting the original polynomials, constituting given cardinal B-spline to the interval $[0, 1]$. For these new polynomials there is a very simple recurrence relation for calculating their coefficients. Moreover, they are much more useful and powerful tools for fast and accurate numerical algorithms.

1. INTRODUCTION

In many problems when a numerical solution of a given differential equation is needed (for example in fluid flow or thermodynamics), the Galerkin B-spline finite element method is applied. This method can be briefly summarized as reformulation of the initial system of differential equations into a system of linear, algebraic equations, where the unknowns are precisely the coefficients of the solution in the B-spline base. To build this linear system, one has to compute the scalar products of the base functions (and products of their derivatives). The accuracy and performance of the Galerkin method depends greatly on this computation. Therefore it is very important to have simple and accurate algorithms for calculating scalar products of basis functions and their derivatives. In the case of cardinal B-splines, formulas for integrals:

$$\int_{\mathbb{R}} N_r^{(m)}(x) N_r^{(n)}(x-k)dx, \quad k \in \mathbb{Z}\,,$$

where N_r denotes the cardinal B-spline of order r and $N_r^{(m)}$ its derivative of order m, are known [2]. Unfortunately, the system of differential equations which is considered, almost always has some boundary conditions, therefore the original basis needs to be modified [4] and consequently, to build the linear system for the Galerkin method, we have to calculate the values of integrals of the form:

$$\int_{l-1}^{l} N_r^{(m)}(x) N_r^{(n)}(x-k)dx\,, \tag{1.1}$$

where $l \in \mathbb{N}$.

Publication co-financed by the European Union as part of the European Social Fund within the project Center for Applications of Mathematics

It is known that cardinal B-splines are constructed from polynomial pieces with rational coefficients. So with a simple and effective formula for those coefficients, we can calculate (1.1) using exact rational arithmetic to improve correctness of numerical methods. In [3], the authors propose an algorithm for calculating the mentioned coefficients, but if r and l are large their method is rather inconvenient to use.

In this paper we propose to consider N_r as a sequence of r polynomials defined on the interval $[0,1]$ and we give a simple algorithm for calculating the coefficients of these new polynomials. By means of these coefficients we can obtain very useful formulas for integrals of the form (1.1).

Let us recall some definitions and basic facts. The cardinal B-spline of order one is the characteristic function of the interval $[0,1)$ and for a natural number $r \geq 2$ we define the cardinal B-spline of order r inductively as a convolution of N_{r-1} and N_1, i.e.

$$N_1(x) = \begin{cases} 1 & \text{if } x \in [0,1), \\ 0 & \text{otherwise} \end{cases}$$

and

$$N_r(x) = N_{r-1} * N_1(x) = \int_{\mathbb{R}} N_{r-1}(x-t) N_1(t) dt = \int_{x-1}^{x} N_{r-1}(t) dt\,.$$

It is well known that the support of N_r is the interval $[0,r]$ and at each interval $[i-1,i]$ for $i \in \{1,2,\ldots,r\}$ the function N_r is a polynomial of degree $r-1$. Moreover, for any $x \in \mathbb{R}$ we have:

$$N_r(x) = \frac{x}{r-1} N_{r-1}(x) + \frac{r-x}{r-1} N_{r-1}(x-1)\,. \tag{1.2}$$

The proof of these facts can be found in [1].

For a given $r \in \mathbb{N}_+$ and $i = 1,\ldots,r$ we define on $[0,1]$ polynomials $p_{r,i}$ which are the original polynomials $N_r|_{[i-1,i]}$ shifted back to 0:

$$p_{r,i}(x) = N_r(x+i-1), \quad x \in [0,1]\,.$$

For example, below we present the explicit formulas for $p_{r,i}$ in case $r = 4$:

$$\begin{aligned} p_{4,1}(x) =& \frac{1}{6}x^3, \\ p_{4,2}(x) =& -\frac{1}{2}x^3 + \frac{1}{2}x^2 + \frac{1}{2}x + \frac{1}{6}, \\ p_{4,3}(x) =& \frac{1}{2}x^3 - x^2 + \frac{2}{3}, \\ p_{4,4}(x) =& -\frac{1}{6}x^3 + \frac{1}{2}x^2 - \frac{1}{2}x + \frac{1}{6}. \end{aligned}$$

We will write $p_{r,i}[k]$ for the coefficient of the monominal x^k in $p_{r,i}(x)$. We assume that for $i \notin \{1,\ldots,r\}$ or $k \notin \{0,\ldots,r-1\}$ the coefficient $p_{r,i}[k]$ is equal to 0.

2. Effective coefficient calculation and formulas for integrals

The theorem presented below forms a foundation of method proposed (please refer [2] for more details). The proof of the theorem, as well as the facts that follow, are very simple, but we present them for the reader's convenience.

Theorem 2.1. *We have $p_{r,i}[k] = \frac{1}{(r-1)!} \cdot q_{r,i}[k]$, where the integers $q_{r,i}[k]$ are defined inductively by $q_{1,1}[0] := 1$ and for $r \in \mathbb{N}_+$, $i \in \{1, \ldots, r+1\}$ and $k \in \{0, \ldots, r\}$*

$$q_{r+1,i}[k] = q_{r,i}[k-1] + (i-1)q_{r,i}[k] + (r-i+2)q_{r,i-1}[k] - q_{r,i-1}[k-1], \tag{2.1}$$

with an additional assumption that $q_{r,i}[k] = 0$ when k or i are outside of defined bounds.

Proof. Let $x \in [0,1]$. Using (1.2) we have:

$$\begin{aligned} q_{r+1,i}(x) &= r! \cdot p_{r+1,i}(x) = r! \cdot N_{r+1}(x+i-1) = \\ &= r! \cdot \left(\frac{x+i-1}{r} N_r(x+i-1) + \frac{r-x-i+2}{r} N_r(x+i-2) \right) = \\ &= r! \cdot \left(\frac{x+i-1}{r} p_{r,i}(x) + \frac{r-x-i+2}{r} p_{r,i-1}(x) \right) = \\ &= (r-1)! \cdot (x \cdot p_{r,i}(x) + (i-1)p_{r,i}(x) + (r-i+2)p_{r,i-1}(x) - x \cdot p_{r,i-1}(x)). \end{aligned}$$

which gives (2.1). □

Remark 2.2. Many relations and symmetries may be found in the system of integers $q_{r,i}[k]$, for example:

$$q_{r,1}[k] = \begin{cases} 0 & \text{if } k \in \{0, \ldots, r-2\}, \\ 1 & \text{if } k = r-1, \end{cases}$$

$$q_{r,2}[k] = \begin{cases} \binom{r-1}{k} & \text{if } k \in \{0, \ldots, r-2\}, \\ -(r-1) & \text{if } k = r-1, \end{cases}$$

$$\sum_{i=1}^{r} q_{r,i}[k] = \begin{cases} (r-1)! & \text{if } k = 0, \\ 1 & \text{if } k \in \{1, \ldots, r-1\}. \end{cases}$$

Moreover, for $i \in \{1, \ldots, r\}$ and $k \in \{0, \ldots, r-2\}$

$$q_{r,i}[k] = (-1)^k q_{r,r+2-i}[k].$$

Applying (2.1) we obtain formulas for the coefficients of the original polynomials, which constitute a cardinal B-spline.

Theorem 2.3. *Let $x \in [s-1, s]$ for some $s \in \{1, \ldots, r\}$ and let $N_r(x) = \sum_{j=0}^{r-1} a_{r,s}[j] x^j$. Then we have:*

$$a_{r,s}[j] = \sum_{k=j}^{r-1} \binom{k}{j} (-s+1)^{k-j} p_{r,s}[k] = \frac{1}{(r-1)!} \sum_{k=j}^{r-1} \binom{k}{j} (-s+1)^{k-j} q_{r,s}[k]\,.$$

Proof. If $x \in [s-1, s]$, then from the definition of $p_{r,s}(x)$:

$$N_r(x) = p_{r,s}(x-s+1) = \sum_{k=0}^{r-1} p_{r,s}[k](x-s+1)^k =$$
$$= \sum_{k=0}^{r-1} p_{r,s}[k] \sum_{j=0}^{k} \binom{k}{j} x^j (-s+1)^{k-j} =$$
$$= \sum_{j=0}^{r-1} \left(\sum_{k=j}^{r-1} \binom{k}{j} (-s+1)^{k-j} p_{r,s}[k] \right) x^j$$

which is our claim. □

Knowing the values of $p_{r,i}[k]$, we can easily obtain derivatives of $p_{r,i}(x)$ of any given order $m \geq 0$. Namely,

$$p_{r,i}^{(m)}(x) = \sum_{k=0}^{r-1-m} \frac{(k+m)!}{k!} p_{r,i}[k+m] x^k =$$
$$= \frac{1}{(r-1)!} \sum_{k=0}^{r-1-m} (k+1)(k+2) \cdot \ldots \cdot (k+m) q_{r,i}[k+m] x^k.$$

For simplicity of notation, for given $r \in \mathbb{N}_+$ and $i \in \{1, \ldots, r\}$ let

$$p_{r,i}^{(m)}[k] = \begin{cases} \frac{(k+m)!}{k!} p_{r,i}[k+m] & \text{if } k \in \{0, \ldots, r-m-1\} \\ 0 & \text{otherwise.} \end{cases}$$

Below, we present simple formulas for calculating integrals of type (1.1).

Theorem 2.4. *Let* $l \in \{1, \ldots, r\}$ *and* $k \in \{0, \ldots, r-1\}$. *For any* $n, m \in \{0, \ldots, r-1\}$ *we have:*

$$\int_{l-1}^{l} N_r^{(m)}(x) N_r^{(n)}(x-k) dx = \sum_{i=0}^{r-1-m} \sum_{j=0}^{r-1-n} \frac{1}{i+j+1} \cdot p_{r,l}^{(m)}[i] \cdot p_{r,l-k}^{(n)}[j].$$

Proof. For l, k, m, n as above we can write:

$$\int_{l-1}^{l} N_r^{(m)}(x) N_r^{(n)}(x-k) dx =$$
$$= \int_0^1 N_r^{(m)}(t+l-1) N_r^{(n)}(t+l-1-k) dt =$$
$$= \int_0^1 p_{r,l}^{(m)}(t) p_{r,l-k}^{(n)}(t) dt =$$
$$= \int_0^1 \left(\sum_{i=0}^{r-1-m} p_{r,l}^{(m)}[i] \cdot t^i \right) \left(\sum_{j=0}^{r-1-n} p_{r,l-k}^{(n)}[j] \cdot t^j \right) dt =$$
$$= \sum_{i=0}^{r-1-m} \sum_{j=0}^{r-1-n} p_{r,l}^{(m)}[i] \cdot p_{r,l-k}^{(n)}[j] \int_0^1 t^{i+j} dt =$$
$$= \sum_{i=0}^{r-1-m} \sum_{j=0}^{r-1-n} p_{r,l}^{(m)}[i] \cdot p_{r,l-k}^{(n)}[j] \cdot \frac{1}{i+j+1},$$

which establishes the formula. □

3. NUMERICAL RESULTS

Let us note, that all of the formulas presented in this article are based on finding $q_{r,i}[k]$, which can be calculated by means of integer operations. After finding $q_{r,i}[k]$ it is reasonable thing to do switch to exact rational representation for calculating and storing $p_{r,i}[k]$ and $a_{r,i}[k]$. When needed, the final results (the resulting finite element matrix for example) can be converted to floating-point numbers.

Implementation of the formulas presented is straightforward. The only important thing to note is that due to the formulas' recursive nature, it is recommended to build a cache for results, and to calculate all coefficients from $r = 1$ up to desired level. This approach requires more memory, but on the other hand it increases performance considerably.

Below we present values of $p_{r,i}[k]$ for $r = 3, \ldots, 7$ calculated by our implementation written in the Python programming language.

$$[p_{3,i}[k]] = \frac{1}{2!}\begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & -2 \\ 1 & -2 & 1 \end{bmatrix},$$

$$[p_{4,i}[k]] = \frac{1}{3!}\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 3 & 3 & -3 \\ 4 & 0 & -6 & 3 \\ 1 & -3 & 3 & -1 \end{bmatrix},$$

$$[p_{5,i}[k]] = \frac{1}{4!}\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 4 & 6 & 4 & -4 \\ 11 & 12 & -6 & -12 & 6 \\ 11 & -12 & -6 & 12 & -4 \\ 1 & -4 & 6 & -4 & 1 \end{bmatrix},$$

$$[p_{6,i}[k]] = \frac{1}{5!}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 5 & 10 & 10 & 5 & -5 \\ 26 & 50 & 20 & -20 & -20 & 10 \\ 66 & 0 & -60 & 0 & 30 & -10 \\ 26 & -50 & 20 & 20 & -20 & 5 \\ 1 & -5 & 10 & -10 & 5 & -1 \end{bmatrix}.$$

$$[p_{7,i}[k]] = \frac{1}{6!}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 6 & 15 & 20 & 15 & 6 & -6 \\ 57 & 150 & 135 & 20 & -45 & -30 & 15 \\ 302 & 240 & -150 & -160 & 30 & 60 & -20 \\ 302 & -240 & -150 & 160 & 30 & -60 & 15 \\ 57 & -150 & 135 & -20 & -45 & 30 & -6 \\ 1 & -6 & 15 & -20 & 15 & -6 & 1 \end{bmatrix},$$

For the sake of comparison, we present the coefficients $a_{7,i}[k]$:

$$[a_{7,i}[k]] = \frac{1}{6!}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -7 & 42 & -105 & 140 & -105 & 42 & -6 \\ 1337 & -3990 & 4935 & -3220 & 1155 & -210 & 15 \\ -24178 & 47040 & -37590 & 15680 & -3570 & 420 & -20 \\ 119182 & -168000 & 96810 & -29120 & 4830 & -420 & 15 \\ -208943 & 225750 & -100065 & 23380 & -3045 & 210 & -6 \\ 117649 & -100842 & 36015 & -6860 & 735 & -42 & 1 \end{bmatrix}.$$

Acknowledgements. Witold Bołt is supported by the Foundation for Polish Science under International PhD Projects in Intelligent Computing. Project financed from The European Union within the Innovative Economy Operational Program 2007-2013 and European Regional Development Fund.

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